MATH–326301

Only approved basic scientific calculators may be used.

This question paper consists of 3 printed pages, each of which is identified by the reference MATH–3263

Examination for the Module MATH–3263
(May–June 2002)

HILBERT SPACES

Time allowed : 3 hours

Do not answer more than **four** questions. All questions carry equal marks.

1. **(a)** Write down the axioms for a complex inner-product space.

State and prove the Cauchy–Schwarz inequality for two vectors x, y in a complex inner-product space. Prove that equality holds in the Cauchy–Schwarz inequality if and only if x and y are linearly dependent.

Show that the following inequality holds for any continuous function f defined on $[0, \pi]$:

$$\left| \int_0^\pi f(t)\sqrt{\sin t}\, dt \right| \leq \sqrt{2} \left(\int_0^\pi |f(t)|^2\, dt \right)^{1/2} .$$

Give a formula for the norm of a vector in an inner-product space E, and, using this definition, prove that the triangle inequality $\|x + y\| \leq \|x\| + \|y\|$ is satisfied for all x, $y \in E$.

(b) Define the complex Hilbert space ℓ_2, giving an expression for the inner product between two vectors in the space.

Let c_{00} denote the subspace of ℓ_2 consisting of all sequences $x = (x_n)_{n=1}^\infty$ such that $x_n = 0$ except for finitely many n. Show that c_{00} is not a closed subspace.

2. **(a)** What is meant by saying that a sequence (e_n) in an inner-product space is *orthonormal*?

Let $\{e_1, \ldots, e_n\}$ be a finite orthonormal set in an inner-product space E and let M be the subspace $\mathrm{Lin}\,\{e_1, \ldots, e_n\}$. Let $x \in E$, and let $y = \sum_{i=1}^n \langle x, e_i\rangle e_i$. Show that $x - y \in M^\perp$. Deduce that $\|x\| \geq \|y\|$, and hence derive Bessel's inequality.

Show also that $\|x - y\| \leq \|x - z\|$ for all $z \in M$.

Show that $\{t, 1-3t^2\}$ is an orthogonal basis for the subspace W of $L_2[-1,1]$ consisting of all polynomials $p(t) = a+bt+ct^2$ such that $\displaystyle\int_{-1}^{1} p(t)\,dt = 0$. Hence determine the best $L_2[-1,1]$ approximation to the function t^3 by an element of W.

(b) State without proof the Riesz–Fischer theorem, which, for an orthonormal sequence (e_n) in a Hilbert space H, characterizes those sequences (λ_n) of complex numbers for which the sum $\sum_{n=1}^{\infty} \lambda_n e_n$ converges in H.

Show that the sequence (e_n), given by $e_n(t) = e^{int}/\sqrt{2\pi}$, is orthonormal in $L_2(-\pi,\pi)$. For which positive values of r is the series $\sum_{n=1}^{\infty} e^{int}/n^r$ the Fourier series of a function in $L_2(-\pi,\pi)$?

3. (a) What is the *orthogonal complement*, $M^\perp$, of a closed linear subspace M of a Hilbert space H?

Prove that $M^\perp$ is a closed linear subspace of H, and that $M \cap M^\perp = \{0\}$.

Let $H = \ell_2$, and let $M = \{(x_n) \in \ell_2 : x_n = 0$ whenever n is a multiple of $3\}$. Determine $M^\perp$, and show directly that $H = M \oplus M^\perp$.

(b) Let X and Y be complex normed spaces. What is a *linear operator* from X to Y? What does it mean to say that such an operator is *bounded*?

Let $B(X,Y)$ denote the space of bounded linear operators from X to Y. Show that $B(X,Y)$ is a normed space with respect to a suitable norm (which should be defined).

Show that the operator $T : \ell_2 \to \ell_2$, defined by $T(x_n) = (y_n)$, where $y_n = n^2 x_n/2^n$ for $n = 1, 2, \ldots$, is bounded, and calculate its norm.

(c) What is the *adjoint*, T^*, of a linear operator T on a Hilbert space H?

Let $T : \ell_2 \to \ell_2$ be the operator defined by

$$T(x_1, x_2, x_3, x_4, \ldots) = (0, 0, x_1, x_2, x_3, \ldots).$$

Give a formula for the operator T^*.

4. (a) Let T be a linear operator on a complex normed space X. What is the definition of the *spectrum*, $\sigma(T)$ of T? What is the *spectral radius*, $r(T)$ of T?

Show that $I - T/\lambda$ is invertible whenever $|\lambda| > \|T\|$, and deduce that $r(T) \le \|T\|$.

Give, without proof, an exact expression for $r(T)$ in terms of the norms of the powers of T.

Question 4 continues . . .

(b) Let T be a linear operator on a complex Hilbert space H. What is meant by the following?

(i) T is *unitary*; **(ii)** T is *Hermitian (self-adjoint)*; **(iii)** T is *normal*.

Show that if T is unitary then T is an isometry on H. Give an example of an isometry which is not unitary.

Let $M : L_2(-1,1) \to L_2(-1,1)$ be defined by $(Mf)(t) = (1-t^2)f(t)$, for $f \in L_2(-1,1)$. Show that M is Hermitian, and that $\sigma(M) = [0,1]$.

5. (a) Let T be a bounded linear operator on a Hilbert space H. What is meant by saying that T is a *Hilbert–Schmidt operator*? What is the *Hilbert–Schmidt norm* of T?

What does it mean to say that an operator T is *compact*?

Let T be a Hilbert–Schmidt operator on H and let (e_n) be an orthonormal basis of H. Show that

$$\left\| T \sum_{n=1}^{\infty} a_n e_n \right\| \leq \left(\sum_{n=1}^{\infty} \|Te_n\|^2 \right)^{1/2} \left\| \sum_{n=1}^{\infty} a_n e_n \right\|.$$

For $k \geq 1$ let T_k denote the operator defined by

$$T_k \left(\sum_{n=1}^{\infty} a_n e_n \right) = \sum_{n=1}^{k} a_n T e_n.$$

Show that $\|T - T_k\| \to 0$, and hence deduce that T is compact.

(b) State (without proof) the spectral theorem for compact normal operators.

Suppose that K is a continuous function on the unit square $[0,1] \times [0,1]$, and let $T : L_2(0,1) \to L_2(0,1)$ be the operator defined by

$$(Tf)(x) = \int_0^1 K(x,y) f(y)\, dy, \qquad \text{for } f \in L_2(0,1).$$

Show that T is a Hilbert–Schmidt operator.

Suppose now that (u_n) is an orthonormal basis of $L_2(0,1)$ consisting of eigenvectors of T, with corresponding eigenvalues (λ_n), and that $1 \notin \sigma(T)$. Show that the Fredholm equation

$$Tf = f + g$$

has a unique solution $f \in L_2(0,1)$ for any $g = \sum_{n=1}^{\infty} \mu_n u_n \in L_2(0,1)$, and express the solution in terms of the eigenvalues and eigenvectors of T.

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