MATH–326301

Only approved basic scientific calculators may be used.

This question paper consists of 3 printed pages, each of which is identified by the reference MATH–3263

Examination for the Module MATH–3263
(May–June 2001)

HILBERT SPACES

Time allowed : 3 hours

Do not answer more than four questions. All questions carry equal marks.

1. (a) What is meant by a *norm* on a complex vector space V?

Let ℓ_1 be the set of complex sequences (x_n) such that $\sum_{n=1}^{\infty} |x_n| < \infty$. Show that ℓ_1 is a vector space, and that defining $\|(x_n)\|_1 = \sum_{n=1}^{\infty} |x_n|$ for $(x_n) \in \ell_1$ makes ℓ_1 into a normed space.

Show that in a complex inner-product space E the following *parallelogram identity* holds for all x, $y \in E$:

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Deduce that the norm on ℓ_1 defined above does not arise from an inner product.

(b) Let Y be a linear subspace of a normed space X. Prove the following:

(i) If Y is complete, then Y is a closed subspace.

(ii) $\overline{Y}$, the closure of Y, is also a linear subspace of X.

Give an example of an *incomplete* inner-product space, and prove that it is incomplete.

2. (a) State and prove the Cauchy–Schwarz inequality for two vectors x, y in a complex inner-product space. Prove that equality holds in the Cauchy–Schwarz inequality if and only if x and y are linearly dependent.

Use the Cauchy–Schwarz inequality to show that

$$\int_0^1 f(t)t^n \, dt \to 0 \qquad \text{as} \quad n \to \infty,$$

for all $f \in C[0, 1]$.

(b) What is meant by saying that a sequence (e_n) in an inner-product space is *orthonormal*?

Let $\{e_1, \ldots, e_n\}$ be a finite orthonormal set in an inner-product space E and let M be the subspace $\mathrm{Lin}\{e_1, \ldots, e_n\}$. Let $x \in E$, and let $y = \sum_{i=1}^{n} \langle x, e_i \rangle e_i$. Show that $x - y \in M^{\perp}$.

Deduce that $\|x\| \geq \|y\|$, and hence derive Bessel's inequality.

Show that the functions e_n for $n \in \mathbb{Z}$, defined by $e_n(t) = \dfrac{1}{\sqrt{2\pi}} e^{int}$, are orthonormal with respect to a certain inner product defined on $C[-\pi, \pi]$, which should be specified.

Deduce that

$$\int_{-\pi}^{\pi} f(t) e^{-int} dt \to 0 \qquad \text{as} \quad n \to \infty,$$

for all $f \in C[-\pi, \pi]$.

3. (a) What is the *orthogonal complement*, $M^{\perp}$, of a closed linear subspace M of a Hilbert space H?

Prove that $M^{\perp}$ is also a closed linear subspace of H, and that $M \cap M^{\perp} = \{0\}$.

Let $H = \ell_2$, and let $M = \{(x_n) \in \ell_2 : x_n = 0 \text{ for } n \text{ even}\}$. Calculate $M^{\perp}$ and show directly that $H = M \oplus M^{\perp}$.

(b) Let X and Y be complex normed spaces. What is a *linear operator* from X to Y? What does it mean to say that such an operator is *bounded*?

Let $B(X, Y)$ denote the space of bounded linear operators from X to Y. Show that $B(X, Y)$ is a normed space with respect to a suitable norm (which should be defined).

Show that the operator $T : \ell_2 \to \ell_2$, defined by $T(x_n) = (y_n)$, where $y_n = 3^n x_n / n!$ for $n = 1, 2, \ldots$, is bounded, and calculate its norm.

(c) What is the *adjoint*, T^*, of a linear operator T on a Hilbert space H?

Let $T : \ell_2 \to \ell_2$ be the operator defined by

$$T(x_1, x_2, x_3, x_4, \ldots) = (x_2, x_3, x_4, \ldots).$$

Give a formula for the operator T^*.

4. (a) Let T be a linear operator on a complex normed space X. What is the definition of the *spectrum*, $\sigma(T)$ of T? What is the *spectral radius*, $r(T)$ of T?

Show that $I - T/\lambda$ is invertible whenever $|\lambda| > \|T\|$, and deduce that $r(T) \leq \|T\|$.

Give, without proof, an exact expression for $r(T)$ in terms of the norms of the powers of T.

 Question 4 continues ...

(b) Let T be a linear operator on a complex Hilbert space H. What is meant by the following?

(i) T is *unitary*; **(ii)** T is *Hermitian (self-adjoint)*; **(iii)** T is *normal*.

Show that if T is unitary then T is an isometry on H. Give an example of an isometry which is not unitary.

Let $M : L_2(-1, 1) \to L_2(-1, 1)$ be defined by $(Mf)(t) = t^2 f(t)$, for $f \in L_2(-1, 1)$. Show that M is Hermitian, and that $\sigma(M) = [0, 1]$.

5. **(a)** Let T be a bounded linear operator on a Hilbert space H. What is meant by saying that T is a *Hilbert–Schmidt operator*?

Let (e_n) and (f_m) be orthonormal bases of H, and T a Hilbert–Schmidt operator defined on H. By expanding each vector Te_n in terms of the basis (f_m) show that

$$\sum_{n=1}^{\infty} \|Te_n\|^2 = \sum_{m=1}^{\infty} \|T^* f_m\|^2.$$

Define the *Hilbert–Schmidt norm* of T with respect to an orthonormal basis (e_n), and show that its definition is independent of the choice of basis.

Show also that $\|T\|_{HS} = \|T^*\|_{HS}$.

Suppose that K is a continuous function on the unit square $[0, 1] \times [0, 1]$, and let $T : L_2(0, 1) \to L_2(0, 1)$ be the operator defined by

$$(Tf)(x) = \int_0^1 K(x, y) f(y)\, dy, \qquad \text{for } f \in L_2(0, 1).$$

Show that T is a Hilbert–Schmidt operator.

(b) What does it mean to say that an operator T is *compact*?

State (without proof) the spectral theorem for compact normal operators.

Use the Neumann series to solve the Volterra integral equation $\phi - \lambda T\phi = f$ in $L_2[0, 1]$, where $\lambda \in \mathbb{C}$, $f(t) = t^2$ for all t, and

$$(T\phi)(x) = 2 \int_0^x t\phi(t)\, dt.$$

END