MATH–321401

This question paper consists of 2 printed pages, each of which is identified by the reference MATH–321401

Only approved basic scientific calculators may be used.

Examination for the Module MATH–3214

(January 2004)

Fourier analysis

Time allowed : 3 hours

Answer not more than **four** questions. All questions carry equal marks.

All functions in this paper are assumed to be Riemann integrable on any finite interval.

1. **(a)** Write down the minimum period, and a fundamental domain, for each of the following functions:

$$(i) \quad \sin x \cos x, \qquad (ii) \quad e^{7ix}.$$

(b) Define the *complex Fourier coefficients* $\hat{f}(n)$, $n \in \mathbb{Z}$, of a 2π-periodic function $f : \mathbb{R} \to \mathbb{C}$. Define also the *Fourier cosine* and *sine coefficients* a_n, b_n of f, and obtain formulas for a_n and b_n in terms of $\hat{f}(n)$ and $\hat{f}(-n)$.

For the function defined by $f(x) = |\sin x|$ $(x \in (-\pi, \pi])$, explain why $b_n = 0$ for all n, and calculate a_n. [You may use the formula $\sin\theta\cos\phi = \frac{1}{2}(\sin(\theta+\phi) + \sin(\theta-\phi))$.]

Deduce that

$$\hat{f}(n) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \frac{-2}{\pi(n^2-1)} & \text{if } n \text{ is even.} \end{cases}$$

2. What does it mean to say that a 2π-periodic function f has an *absolutely convergent Fourier series*?

Show that if f has a continuous derivative then $\widehat{f'}(n) = in\hat{f}(n)$. Deduce that if f has a continuous second derivative then f has an absolutely convergent Fourier series.

Assuming that the Fourier series of f converges uniformly to f, and that g is another 2π-periodic function with absolutely convergent Fourier series, find an expression for the nth Fourier coefficient of the product fg, and deduce that fg also has absolutely convergent Fourier series.

Prove that $|f(x)| \leqslant \sum_{n\in\mathbb{Z}} |\hat{f}(n)|$ for all x.

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3. **(a)** Define the *convolution* $f * g$ of two continuous 2π-periodic functions $f,\ g : \mathbb{R} \to \mathbb{C}$. Show that if g is a trigonometric polynomial then so is $f * g$.

Define the *Dirichlet kernel functions* $(D_k)_{k=0}^{\infty}$, and show that the partial sums $s_k(f)$ of the Fourier series of f satisfy $s_k(f) = f * D_k$.

(b) State (without proof) Parseval's identity.

For a fixed $\lambda > 0$, let f be the 2π-periodic function defined on $(-\pi, \pi]$ by $f(x) = e^{\lambda x}$. Calculate the Fourier coefficients of f, and apply Parseval's identity to show that

$$\sum_{n=-\infty}^{\infty} \frac{1}{\lambda^2 + n^2} = \frac{\pi \cosh \lambda\pi}{\lambda \sinh \lambda\pi}.$$

4. **(a)** Define the *Fourier transform* $\hat{f}$ of a function $f : \mathbb{R} \to \mathbb{C}$ satisfying $\int_{\mathbb{R}} |f(x)|\, dx < \infty$. Show that $\hat{f}$ is bounded, with $|\hat{f}(w)| \leqslant \int_{\mathbb{R}} |f(x)|\, dx$.

State and prove the Riemann–Lebesgue Lemma for Fourier transforms.

(b) Find the Fourier transform of the function f defined by

$$f(x) = \begin{cases} x & \text{if } |x| \leqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

Verify that $\hat{f}$ is continuous at $x = 0$.

5. **(a)** Define the *inverse Fourier transform* $\check{f}$ of a function $f : \mathbb{R} \to \mathbb{C}$ satisfying $\int_{\mathbb{R}} |f(w)|\, dw < \infty$.

Give a careful statement (without proof) of Fourier's Inversion Theorem.

(b) Define the *Schwartz class* $\mathcal{S}$ of rapidly decreasing functions on $\mathbb{R}$. Give an example of a nonzero function in $\mathcal{S}$ which vanishes outside the interval $[-1, 1]$. [You are **not** asked to explain why your function is in $\mathcal{S}$.]

Prove that if $f \in \mathcal{S}$ then also $\hat{f} \in \mathcal{S}$. [You may assume a theorem about differentiating under the integral sign, provided that it is clearly stated.]

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