This question paper consists of 2 printed pages, each of which is identified by the reference MATH-318101.

MATH-318101

Only approved basic scientific calculators may be used.

Examination for the Module MATH-3181
(January 2007)

Inner-product and Metric Spaces

Time allowed : 2 hours

Do not attempt more than **four** questions
All questions carry equal marks

1. **(a)** Let V be a vector space over the real numbers. State what is meant by saying that $\langle\ ,\ \rangle$ defines an *inner product* on V.

Which of the following defines an inner product on $\mathbb{R}^2$? Give proofs or counterexamples as appropriate. (We write $\mathbf{v} = (v_1, v_2)$ and $\mathbf{w} = (w_1, w_2)$.)

(i) $\langle \mathbf{v}, \mathbf{w} \rangle = 2v_1w_1 + v_1w_2 + v_2w_1 + 2v_2w_2$; **(ii)** $\langle \mathbf{v}, \mathbf{w} \rangle = v_1w_2$;
(iii) $\langle \mathbf{v}, \mathbf{w} \rangle = v_1^2w_1^2 + v_2w_2$.

(b) Let V be an inner-product space with inner product $\langle\ ,\ \rangle$. Define the *norm*, $\|\mathbf{v}\|$, of an element $\mathbf{v} \in V$, and show that we have $|\langle \mathbf{v}, \mathbf{w} \rangle| \le \|\mathbf{v}\|\,\|\mathbf{w}\|$ for all $\mathbf{v}$, $\mathbf{w} \in V$.

What is meant by the *angle*, θ, between two nonzero vectors $\mathbf{v}$ and $\mathbf{w}$?

Suppose that $\|\mathbf{v}\| = 1$ and $\|\mathbf{w}\| = 2$; find a formula for $\|\mathbf{v} + \mathbf{w}\|$ in terms of θ. Calculate θ given that $\|\mathbf{v} + \mathbf{w}\| = \sqrt{3}$.

2. **(a)** Define the sequence space ℓ^2, and give a formula for the inner-product between two sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$.

Which of the following sequences lies in ℓ^2? **(i)** $\left(\frac{1}{n}\right)_{n=1}^{\infty}$; **(ii)** $\left(\frac{1}{\sqrt{n}}\right)_{n=1}^{\infty}$.

Calculate the ℓ^2 norm of the sequence $(r^n)_{n=1}^{\infty}$, where r is a real number with $|r| < 1$.

(b) What is meant by saying that a set $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ in an inner-product space V is *orthonormal*?

Prove that every orthonormal set $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is linearly independent.

(c) Apply the Gram–Schmidt procedure to find an orthonormal basis for the subspace of $\mathbb{R}^4$ (with the standard inner product) spanned by the vectors $(0, 0, 1, 1)$, $(1, 2, 3, 1)$ and $(3, 1, 2, 0)$.

3. **(a)** Let V be an inner-product space and let W be a subspace of V. Suppose that for $\mathbf{v} \in V$ and $\mathbf{w} \in W$ the condition $\langle \mathbf{v} - \mathbf{w}, \mathbf{x} \rangle = 0$ holds for all $\mathbf{x} \in W$. Show that $\|\mathbf{v} - \mathbf{w}\| \leq \|\mathbf{v} - \mathbf{y}\|$ for all $\mathbf{y} \in W$.

Suppose further that W has basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ (not necessarily orthonormal), and that $\mathbf{w} = a_1\mathbf{w}_1 + \ldots + a_n\mathbf{w}_n$. Derive the *normal equations* satisfied by $a_1, \ldots, a_n$.

(b) A quantity z should theoretically depend on the variables x and y by means of the formula $z = a + bxy + cy^2$. Find the values of the constants a, b and c so that the formula best fits the following experimental data, in the sense of least squares approximation:

	x	y	z
First measurement	0	1	4
Second measurement	0	2	9
Third measurement	1	-1	5
Fourth measurement	1	1	9

4. **(a)** State the axioms for a metric space.

Define the space $C[0,1]$ and give the formulae for the three standard metrics d_1, d_2 and d_∞ defined on it (there is no need to prove that these satisfy the axioms for a metric).

Calculate the distance, in each of the metrics d_1 and d_∞, between the functions $f(x) = 2x$ and $g(x) = 1 - x$.

(b) What is meant by saying that a sequence (x_n) in a metric space (X, d) is (i) *convergent*, and (ii) *Cauchy?*

Give the definition of a *complete* metric space. Which of the following subsets of $\mathbb{R}$ is complete in the usual metric $d(x, y) = |x - y|$? Give brief explanations.

(i) $(0, 1)$; **(ii)** $[0, 1]$; **(iii)** $\mathbb{Q}$; **(iv)** $\{0\}$.

5. **(a)** Let $\phi : X \to X$ be a mapping on a metric space (X, d). Define what is meant by saying that ϕ is a *contraction mapping* and state the *Contraction Mapping Theorem.*

(b) Show that the mapping ϕ defined by

$$\phi(f)(x) = 3 + \int_0^x (2\cos t + t^2 f(t))\,dt$$

is a contraction mapping on $C[0,1]$, when the space is given an appropriate metric, which you should specify.

Using the contraction mapping theorem, deduce that the differential equation

$$\frac{dy}{dx} = 2\cos x + x^2 y$$

has a unique solution in $C[0,1]$ such that $y = 3$ when $x = 0$.

Starting with the initial approximate solution $y = f_0(x) \equiv 3$, find a closer approximation to the actual solution.

END