This question paper consists of 2 printed pages, each of which is identified by the reference MATH-318101.

MATH-318101

Only approved basic scientific calculators may be used.

Examination for the Module MATH-3181
(January 2006)

Inner-product and Metric Spaces

Time allowed : 2 hours

Do not attempt more than **four** questions
All questions carry equal marks

1. **(a)** Let V be a vector space over the real numbers. State what is meant by saying that $\langle\ ,\ \rangle$ defines an *inner product* on V.

Let $C[0,1]$ denote the space of continuous real-valued functions on the closed interval $[0,1]$ of the real line. Which of the following defines an inner product on $C[0,1]$? Give proofs or counterexamples as appropriate.

(i) $\langle f,g\rangle = \displaystyle\int_0^1 (1-2t)f(t)g(t)\,dt$; (ii) $\langle f,g\rangle = \displaystyle\int_0^1 (f(t)+g(t))\,dt$;

(iii) $\langle f,g\rangle = \displaystyle\int_0^1 t^2 f(t)g(t)\,dt$.

(b) Let V be an inner-product space with inner product $\langle\ ,\ \rangle$. Define the *norm*, $\|\mathbf{v}\|$, of an element $\mathbf{v}\in V$, and show that we have $|\langle\mathbf{v},\mathbf{w}\rangle| \le \|\mathbf{v}\|\,\|\mathbf{w}\|$ for all $\mathbf{v}$, $\mathbf{w}\in V$.

Deduce that $\|\mathbf{v}+\mathbf{w}\| \le \|\mathbf{v}\| + \|\mathbf{w}\|$.

Given two vectors $\mathbf{v}$ and $\mathbf{w}$ with $\|\mathbf{v}+\mathbf{w}\| = 2$ and $\|\mathbf{v}-\mathbf{w}\| = 1$, find $\langle\mathbf{v},\mathbf{w}\rangle$.

2. **(a)** Let V be an inner-product space with inner product $\langle\ ,\ \rangle$. What is meant by saying that a set $\{\mathbf{v}_1,\ldots,\mathbf{v}_n\}$ of vectors in V is *orthogonal*?

Let $C[-1,1]$ denote the space of continuous functions on the interval $[-1,1]$ with inner product given by $\langle f,g\rangle = \displaystyle\int_{-1}^1 f(t)g(t)\,dt$. Find real constants a, b and c such that the functions 1, $t+a$ and t^2+bt+c form an orthogonal set in $C[-1,1]$.

(b) What is meant by saying that a set $\{\mathbf{e}_1,\ldots,\mathbf{e}_n\}$ in an inner-product space V is *orthonormal*? Let $\mathbf{v}$ be an element of V and $\{\mathbf{e}_1,\ldots,\mathbf{e}_n\}$ an orthonormal basis of V. Write down an expression for the coefficients c_k such that $\mathbf{v} = \sum_{k=1}^n c_k\mathbf{e}_k$.

(c) Apply the Gram–Schmidt procedure to find an orthonormal basis for the subspace of $\mathbb{R}^4$ (with the standard inner product) spanned by the vectors $(1,0,0,1)$, $(4,0,1,2)$ and $(3,3,3,-3)$.

Find the orthogonal projection of the vector $(0, 0, 1, 1)$ on W.

3. **(a)** Let V be an inner-product space and let W be a subspace of V. Suppose that for $\mathbf{v} \in V$ and $\mathbf{w} \in W$ the condition $\langle \mathbf{v} - \mathbf{w}, \mathbf{x} \rangle = 0$ holds for all $\mathbf{x} \in W$. Show that $\|\mathbf{v} - \mathbf{w}\| \leq \|\mathbf{v} - \mathbf{y}\|$ for all $\mathbf{y} \in W$.

Suppose further that W has basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ (not necessarily orthonormal), and that $\mathbf{w} = c_1\mathbf{w}_1 + \ldots + c_n\mathbf{w}_n$. Derive the *normal equations* satisfied by $c_1, \ldots, c_n$.

(b) A quantity z should theoretically depend on the variables x and y by means of the formula $z = ax^2 + bxy + cy^2$. Find the values of the constants a, b and c so that the formula best fits the following experimental data, in the sense of least squares approximation:

	x	y	z
First measurement	1	0	-2
Second measurement	2	0	9
Third measurement	1	1	6
Fourth measurement	0	1	1

4. **(a)** State the axioms for a metric space.

Show that $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for every $a, b \geq 0$, and hence prove that $(\mathbb{R}, d)$ is a metric space, where d is defined by $d(x, y) = |x - y|^{1/2}$.

(b) What is meant by saying that a sequence (x_n) in a metric space (X, d) is (i) *convergent*, and (ii) *Cauchy?*

Give the definition of a *complete* metric space, and show that the open interval $(0, 1)$ is not complete under the usual metric $d(x, y) = |x - y|$.

(c) Give the definition of the *discrete metric* d_0 on an arbitrary nonempty set X.

Show that a sequence (x_n) is a Cauchy sequence in (X, d_0) if and only if for some N (depending on the sequence) $x_N = x_{N+1} = x_{N+2} = \cdots$. Deduce that X is complete under the discrete metric.

5. **(a)** Let $\phi : X \to X$ be a mapping on a metric space (X, d). Define what is meant by saying that ϕ is a *contraction mapping.*

(b) State and prove the *Contraction Mapping Theorem* for a complete metric space (X, d).

(c) Show that $\phi(x) = (x^4 + x^3 + 5)/8$ is a contraction mapping on $[-1, 1]$.

Let $x_0 = 0$, and define $x_{n+1} = \phi(x_n)$ for each $n \geq 0$. Explain why the sequence (x_n) necessarily converges, and write down a polynomial equation satisfied by its limit L.

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