MATH-318101

Only approved basic scientific calculators may be used.

This question paper consists of 2 printed pages, each of which is identified by the reference MATH-318101.

Examination for the Module MATH-3181
(January 2005)

Inner-product and Metric Spaces

Time allowed : 2 hours

Do not attempt more than **four** questions
All questions carry equal marks

1. **(a)** Let V be a vector space over the real numbers. State what is meant by saying that $\langle\ ,\ \rangle$ defines an *inner product* on V.

Which of the following defines an inner product on $\mathbb{R}^2$? Give proofs or counter-examples, as appropriate. (We write $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$.)

(i) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 - x_2y_2$; (ii) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1x_2 + y_1y_2$; (iii) $\langle \mathbf{x}, \mathbf{y} \rangle = 2x_1y_1 + 3x_2y_2$.

(b) Let V be an inner-product space with inner product $\langle\ ,\ \rangle$. Define the *norm*, $\|\mathbf{v}\|$, of an element $\mathbf{v} \in V$, and show that we have $|\langle \mathbf{v}, \mathbf{w} \rangle| \le \|\mathbf{v}\|\,\|\mathbf{w}\|$ for all $\mathbf{v}$, $\mathbf{w} \in V$.

By applying this inequality in the space $V = C[0, 1]$ with a suitable inner product, prove that

$$\int_0^1 \frac{e^x}{x+1}\,dx \le \left(\frac{e^2-1}{4}\right)^{1/2}.$$

2. **(a)** Let V be an inner-product space with inner product $\langle\ ,\ \rangle$. What is meant by the *angle* between two non-zero vectors $\mathbf{v}$ and $\mathbf{w}$ in V?

Suppose that $\mathbf{v}$ and $\mathbf{w}$ are vectors in V such that $\|\mathbf{v}\| = \|\mathbf{w}\| = 1$ and the angle between $\mathbf{v}$ and $\mathbf{w}$ is $\pi/4$. Calculate $\langle \mathbf{v}, \mathbf{w} \rangle$ and hence find $\|\mathbf{v} + \sqrt{2}\mathbf{w}\|$.

What is meant by saying that $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is an *orthonormal basis* of V? Let $\mathbf{v}$ be an element of V and $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ an orthonormal basis of V. Write down an expression for the coefficients c_k such that $\mathbf{v} = \sum_{k=1}^{n} c_k\mathbf{e}_k$.

(b) Apply the Gram–Schmidt procedure to find an orthonormal basis for the subspace W of $\mathbb{R}^4$ (with the standard inner product) spanned by the vectors $(1, 2, 0, 2)$, $(1, 3, 1, 1)$ and $(3, 2, 2, 1)$.

Find the orthogonal projection of the vector $(0, 0, 0, 1)$ on W.

continued ...

3. **(a)** Let V be an inner-product space and let W be a subspace of V. Suppose that for $\mathbf{v} \in V$ and $\mathbf{w} \in W$ the condition $\langle \mathbf{v} - \mathbf{w}, \mathbf{x} \rangle = 0$ holds for all $\mathbf{x} \in W$. Show that $\|\mathbf{v} - \mathbf{w}\| \leq \|\mathbf{v} - \mathbf{y}\|$ for all $\mathbf{y} \in W$.

Suppose further that W has basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ (not necessarily orthonormal), and that $\mathbf{w} = c_1\mathbf{w}_1 + \ldots + c_n\mathbf{w}_n$. Derive the *normal equations* satisfied by $c_1, \ldots, c_n$.

(b) A quantity z should theoretically depend on the variables x and y by means of the formula $z = ax + by + cxy$. Find the values of the constants a, b and c so that the formula best fits the following experimental data, in the sense of least squares approximation:

	x	y	z
First measurement	1	0	2.0
Second measurement	0	1	3.0
Third measurement	1	1	3.0
Fourth measurement	2	0	1.5

4. **(a)** State the axioms for a metric space.

For functions $f, g \in C[0,2]$ define

$$d_1(f,g) = \int_0^2 |f(t) - g(t)|\, dt.$$

Show that d_1 is a metric on $C[0,2]$.

(b) What is meant by saying that a sequence (x_n) in a metric space (X,d) is (i) *convergent*, and (ii) *Cauchy?* Prove that every convergent sequence is Cauchy.

Give the definition of a *complete* metric space.

Let (f_n) be the sequence of functions in $C[0,2]$ given by

$$f_n(t) = \begin{cases} t^n & \text{if } 0 \leq t \leq 1, \\ 1 & \text{if } 1 \leq t \leq 2. \end{cases}$$

Show that (f_n) is a Cauchy sequence in the metric space $(C[0,2], d_1)$. Suppose now that (f_n) tends to a limit $f \in C[0,2]$. Calculate $\int_0^1 |f(t)|\, dt$ and $\int_1^2 |1 - f(t)|\, dt$, and deduce that $(C[0,2], d_1)$ is incomplete.

5. **(a)** Let $\phi : X \to X$ be a mapping on a metric space (X,d). Define what is meant by saying that ϕ is a *contraction mapping.*

(b) State and prove the *Contraction Mapping Theorem* for a complete metric space (X,d).

(c) Show that the equation

$$x^4 - x^2 + 5 = 7x$$

has precisely one solution in the interval $[-1,1]$.

END