MATH-318101

This question paper consists of 3 printed pages, each of which is identified by the reference MATH-318101.

Only approved basic scientific calculators may be used

Examination for the Module MATH3181
(January 2004)

Inner Product and Metric Spaces
Time allowed : 2 hours

Answer no more than **FOUR** questions.

1. (i) Let X be a vector space over the real numbers. State what is meant by saying that $\langle\ ,\ \rangle$ is an *inner-product* on X.
Show that neither
(*a*)
$$\langle \mathbf{x}, \mathbf{y}\rangle := x_1 y_1$$
nor
(*b*)
$$\langle \mathbf{x}, \mathbf{y}\rangle := x_1^2 y_1^2 + x_2^2 y_2^2$$
define inner products on $\mathbb{R}^2$. (Here $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$.)

(ii) Let $(X, \langle\ ,\ \rangle)$ be an inner-product space. Define the *norm* $\|\mathbf{x}\|$ of an element $\mathbf{x} \in X$, and show that, for all $\mathbf{x}, \mathbf{y} \in X$, we have
$$|\langle \mathbf{x}, \mathbf{y}\rangle| \leq \|\mathbf{x}\|\,\|\mathbf{y}\|.$$
By applying this inequality with a suitable inner product on $X = C[\pi/2, \pi]$ (the space of continuous functions on the interval $[\pi/2, \pi]$), show that
$$\int_{\pi/2}^{\pi} \frac{\sin t}{t}\, dt \leq \frac{1}{\sqrt{\pi}} \left\{ \int_{\pi/2}^{\pi} \sin^2 t \right\}^{1/2}.$$

2. (i) Let $(X, \langle\ ,\ \rangle)$ be an inner-product space. Show that, for any pair of elements $\mathbf{x}, \mathbf{y} \in X$ satisfying $\langle \mathbf{x}, \mathbf{y}\rangle = 0$, we have
$$\|\mathbf{x} - \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2.$$

(ii) Let $(X, \langle\ ,\ \rangle)$ be an inner-product space. What is meant by saying that $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is an *orthonormal* set in X?
Let E be a subspace of X and let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be an orthonormal basis of E. Show that, for $\mathbf{x} \in E$,
$$\mathbf{x} = \sum_{j=1}^{n} \langle \mathbf{x}, \mathbf{e}_j\rangle \mathbf{e}_j.$$

(iii) Let $\mathbb{R}^4$ have the usual inner-product, given by
$$\langle \mathbf{x}, \mathbf{y}\rangle = \sum_{j=1}^{4} x_j y_j.$$
Find an orthonormal basis of the subspace E spanned by the set of vectors
$$\{(1,1,1,1), (1,0,1,0), (1,1,0,1)\}$$
Find the orthogonal projection of $(1,1,0,0)$ on E

continued

3. (i) $(X, \langle\ ,\ \rangle)$ be an inner product space, and let E be a subspace of X. Suppose that $\mathbf{y} \in X$ and $\mathbf{u} \in E$ satisfy the condition

$$\langle \mathbf{y} - \mathbf{u}, \mathbf{x} \rangle = 0 \qquad \text{for all } \mathbf{x} \in E.$$

Show that $\|\mathbf{y} - \mathbf{u}\| \leq \|\mathbf{y} - \mathbf{w}\|$ for all $\mathbf{w} \in E$.

(ii) Let $(X, \langle\ ,\ \rangle)$ be an inner product space, and let E be a subspace with a basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$. Let $\mathbf{y} \in X$, and let $P_E(\mathbf{y}) = k_1\mathbf{x}_1 + \cdots + k_n\mathbf{x}_n$ be the orthogonal projection of $\mathbf{y}$ on E. Derive the *normal equations* for the real numbers $k_1, \ldots, k_n$

(ii) A quantity y should theoretically depend on variables s and t by means of a formula $y = a + bs^2 + ct$. Find the values of the constants a, b, and c so that the formula best fits the following experimental data (in the sense of least squares approximation).

	s	t	y
First measurement:	0	1	5.5
Second measurement:	0	−1	−0.5
Third measurement:	2	1	10
Fourth measurement:	−2	−1	3

4. (i) Let (X, d) be a metric space and let (x_n) be a sequence in X. State precisely what is meant by saying that:

(a) $x_n \to x \in X$, as $n \to \infty$;
(b) (x_n) is a *Cauchy sequence*;
(c) (X, d) is *complete*.

(ii) Prove that, if $x_n \to x$ as $n \to \infty$, then (x_n) is a Cauchy sequence.

(iii) Let X be $C[0,1]$, the space of continuous functions on the interval $[0,1]$. Let the metric d_∞ on X be given by

$$d_\infty(f,g) := \max_{0 \leq t \leq 1} |f(t) - g(t)|$$

for $f, g \in X$

Let f be the function which is identically zero. Show that $f_n \to f$ for d_∞, when f_n is defined by

$$f_n(t) := \frac{t^n}{n}.$$

Show that (g_n) does not converge to f in d_∞ when g_n is defined by

$$g_n(t) := t^n(1 - t^n)$$

Hence or otherwise show that (g_n) is not a Cauchy sequence for d_∞.

continued ...

5. (i) Define what is meant by a *contraction* mapping on a metric space (X, d) and state the Contraction Mapping Theorem.

(ii) Let the diferential equation

$$\frac{dy}{dt} = F(t, y) \qquad \text{with initial condition } y(a) = c$$

satisfy the conditions F is continuous and

$$|F(t, y_1) - F(t, y_2)| \leq \frac{k}{b-a}|y_1 - y_2|$$

for all $t \in [a, b]$ and all $y_1, y_2 \in \mathbb{R}$. Show that there is a unique function $f \in C[a, b]$ such that f is a solution of the differential equation with the given initial condition Hence show that the differential equation

$$\frac{dy}{dt} = t \sin t - t^2 y \qquad \text{with initial condition } y(0) = 1$$

has precisely one solution on any interval $[0, b]$, where $0 < b < 1$.

END