MATH-318101

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Examination for the Module MATH3181
(May-June 2003)

Inner Product and Metric Spaces

Time allowed : 2 hours

Answer no more than **FOUR** questions.

1. (i) Let X be a vector space over the real numbers. State what is meant by saying that $\langle\ ,\ \rangle$ is an *inner product* on X.
Let $C[0,1]$ denote the space of continuous real-valued functions on the closed interval $[0,1]$ of the real line.
Show that

$$\langle f,g\rangle := \int_0^{1/2} f(t)g(t)\,dt$$

is not an inner product on $C[0,1]$.

(ii) Let $(X, \langle\ ,\ \rangle)$ be an inner-product space. Define the *norm* $\|\mathbf{x}\|$ of an element $\mathbf{x}\in X$, and show that, for all $\mathbf{x},\mathbf{y}\in X$, we have

$$|\langle \mathbf{x},\mathbf{y}\rangle| \le \|\mathbf{x}\|\,\|\mathbf{y}\|.$$

Deduce that $\|\mathbf{x}+\mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$.
Let $\mathbb{R}^2$ have the standard inner product, given by

$$\langle \mathbf{x},\mathbf{y}\rangle := x_1y_1 + x_2y_2 .$$

Find elements $\mathbf{x},\mathbf{y}\in\mathbb{R}^2$ such that

$$\|\mathbf{x}+\mathbf{y}\|^2 > \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 .$$

2. Let $(X, \langle\ ,\ \rangle)$ be an inner-product space.

(i) Show that, for any $\mathbf{x},\mathbf{y}\in X$,

$$\|2\mathbf{x}+3\mathbf{y}\|^2 + \|3\mathbf{x}-2\mathbf{y}\|^2 = 13\left(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2\right)$$

(ii) State what is meant by saying that elements $\mathbf{x},\mathbf{y}\in X$ are *orthogonal*.
Show that, for any pair of orthogonal elements $\mathbf{x},\mathbf{y}\in X$, we have

$$\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 .$$

and deduce that, if $\mathbf{z}\in X$ is also orthogonal to both $\mathbf{x}$ and $\mathbf{y}$, then

$$\|\mathbf{x}+\mathbf{y}+2\mathbf{z}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 4\|\mathbf{z}\|^2.$$

(iii) Let $C[-1,1]$ be the space of continuous real-valued functions on the closed interval $[-1,1]$ of the real line. With the inner product $\langle f,g\rangle := \int_{-1}^{1} f(t)g(t)\,dt$, show that the functions f and g defined by $f(t) :\equiv 1$ and $g(t) := t$ are orthogonal. Find values of a and b such that the function h defined by $h(t) := t^2 + at + b$ is orthogonal to f and g.

continued . .

3. (i) Let $(X, \langle\ ,\ \rangle)$ be an inner product space, let E be a vector subspace, and let $\mathbf{y} \in X$. Let $P_E(\mathbf{y}) \in E$ satisfy $\langle \mathbf{y} - P_E(\mathbf{y}), \mathbf{x} \rangle = 0$ for all $\mathbf{x} \in E$. Show that, if $\mathbf{y} \in E$, then $P_E(\mathbf{y}) = \mathbf{y}$.

(ii) Let $(X, \langle\ ,\ \rangle)$ be an inner product space and let E be a subspace with a basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$. Let $\mathbf{y} \in X$, and let $P_E(\mathbf{y}) = a_1\mathbf{x}_1 + \cdots + a_n\mathbf{x}_n$. Derive the *normal equations* for the real numbers $a_1, \ldots, a_n$.

Deduce that if the basis is orthonormal, then $P_E(\mathbf{y}) = \sum_{j=1}^{n} \langle \mathbf{y}, \mathbf{x}_j \rangle \mathbf{x}_j$.

(iii) A quantity y should theoretically depend on a variable x by means of a formula $y = a + bx + cx^2$. Find the values of the constants a, b, and c, so that the formula best fits the following experimental data (in the sense of least squares approximation)

	x	y
First measurement:	1	3.2
Second measurement:	−1	1.1
Third measurement:	2	2.8
Fourth measurement:	−2	6.9

4. Let (X, d) be a metric space and let (x_n) be a sequence in X. State precisely what is meant by saying that:

(a) $x_n \to x \in X$, as $n \to \infty$;

(b) (x_n) is a *Cauchy sequence*;

(c) (X, d) is *complete*.

(i) Prove that, if $x_n \to x$ as $n \to \infty$, then (x_n) is a Cauchy sequence.

(ii) Prove also that if $x_n \to x$ and $x_n \to y$ as $n \to \infty$, then $x = y$.

(iii) Let $\mathbb{R}^2$ have the metric $d_1(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + |x_2 - y_2|$. Show that if the sequence $\mathbf{x}_n \to \mathbf{x}$ then the two sequences of co-ordinates $(x_{n,1})$ and $(x_{n,2})$ converge in $\mathbb{R}$ (with the standard metric).

(iv) Let X be an arbitrary set and let d be the discrete metric on X defined by

$$d(x, y) := \begin{cases} 1, & \text{if } x \neq y; \\ 0, & \text{if } x = y. \end{cases}$$

Show that (X, d) is complete.

5. (i) Let $\phi : X \to X$ be a mapping on a metric space (X, d). Define what is meant by saying that ϕ is a *contraction mapping*.

(ii) State and prove the *Contraction Mapping Theorem* for a complete metric space (X, d)

(iii) Show that the equation

$$x^4 - 2x^3 + 2 = 3x$$

has precisely one solution in the interval $[0, 1]$.

END