MATH-316301

This question paper consists of 3 printed pages, each of which is identified by the reference MATH 316301

No calculators allowed

Examination for the Module MATH 3163
(January 2005)

COMPUTABILITY AND UNSOLVABILITY

Time allowed : 3 hours

Do not answer more than *FOUR* questions.
All questions carry equal marks.

1. (**a**) (i) Show that $f(m,n) = m+n$, and

$$\overline{sg}(n) = \begin{cases} 0 & \text{if } n > 0, \\ 1 & \text{if } n = 0, \end{cases}$$

are primitive recursive functions.

(ii) Show that if given sets A and B are primitive recursive, then so is their set theoretic union $A \cup B$.

(**b**) Write Turing programs for $f(m,n) = m+n$ and $\overline{sg}(n)$, and briefly explain why each program works.

(**c**) Describe *briefly* how it is possible to effectively list all Turing programs:

$$P_0, P_1, \ldots, P_e, \ldots .$$

Prove the *Enumeration Theorem:* There exists a partial computable function $\varphi_z(x)$ (of x and z) such that, for each $e \in \mathbb{N}$, φ_e is the e^{th} (unary) partial computable function.

Deduce that there exists a *Universal Turing Machine U* which, if given input (e,x), simulates the e^{th} Turing machine with input x (that is, $\varphi_U^{(2)}(e,x) = \varphi_e(x)$ for all $x \in \mathbb{N}$).

 continued ...

2. (**a**) For each $e \in \mathbb{N}$, we define computable approximations for φ_e (the e^{th} partial computable function in some standard listing) by:

$$\varphi_{e,s}(x) = y \iff_{\text{defn.}} \begin{array}{l} x, y, e < s, \text{ and } y \text{ is the output of } \varphi_e(x) \\ \text{in } < s \text{ steps of the Turing program } P_e. \end{array}$$

Show that $\varphi_{e,s}(x)\downarrow$ ("$\varphi_{e,s}(x)$ is defined") is a computable relation over $e, s, x \in \mathbb{N}$,

(**b**) Define what is meant by saying that a set A is *computably enumerable*.

Prove the *Normal Form Theorem* giving the equivalence of the following three statements:

i) A is computably enumerable,

ii) A is a Σ^0_1 set,

iii) A is the domain of some p.c. function φ_e.

(**c**) Show that:

(i) $\varphi_x(x)\downarrow$ is a Σ^0_1 relation, but

(ii) $\varphi_x(x)\downarrow$ is not a computable relation.

Deduce that there exists a c.e. set which is not computable.

[You may assume in part (c) that if a set A is computable, then it is also computably enumerable.]

3. (**a**) Define: $A \subseteq \mathbb{N}$ is *simple*.

(i) Show that no simple set is computable.

(ii) Show that if A is a simple set and W is an infinite c.e. set then $A \cap W$ is an infinite c.e. set.

(iii) Deduce that if A and B are simple sets then $A \cap B$ is also simple.

[You may assume that the intersection $X \cap Y$ of two computably enumerable sets X and Y is also computably enumerable.]

(**b**) For $A, B, C \subseteq \mathbb{N}$ define : $A \leq_m B$ (A is *many-one reducible* to B); C is *creative*.

(i) Prove that if A is a c.e. set and ψ is a partial computable function, then $\psi^{-1}(A)$ is also c.e.

(ii) Show that if C is creative, and A is a c.e. set such that $C \leq_m A$, then A is also creative.

 continued . . .

4. (**a**) Define what is meant by the *m-degree* of a set $A \subseteq \mathbb{N}$, and define the partial ordering $\leq$ on the set of all m-degrees.

(i) Let $\{W_i\}_{i \in \mathbb{N}}$ be the usual listing of the computably enumerable sets.

Show that $A \subseteq \mathbb{N}$ is computably enumerable if and only if $A \leq_m K_0$, where K_0 is defined by

$$K_0 = \{\langle x, y\rangle \mid x \in W_y\}.$$

(ii) Let $\mathbf{0}'_m = \deg_m(K_0)$.

Show that the following are equivalent:

(a) $\mathbf{a}_m \leq \mathbf{0}'_m$.

(b) $\mathbf{a}_m$ is computably enumerable.

(c) Every $A \in \mathbf{a}_m$ is computably enumerable.

Deduce that $\mathbf{0}'_m > \mathbf{0}_m$, and $\mathbf{0}'_m$ is the *greatest* computably enumerable m-degree.

(**b**) Define the notions $A \leq_T B$ (that is, A is *Turing reducible to* B), and $A \equiv_T B$ (that is, A is *Turing equivalent to* B), where $A, B \subseteq \mathbb{N}$.

Let $\mathbf{a}$ be a Turing degree.

(i) Show that $\{X \subseteq \mathbb{N} \mid X \in \mathbf{a}\}$ is a countably infinite set.

(ii) Show that $\{\mathbf{b} \mid \mathbf{b} \leq \mathbf{a}\}$ is countable.

(iii) Show, however, that $\boldsymbol{\mathcal{D}}$ (= the set of all Turing degrees) is uncountable.

Deduce that there is no greatest member of $\boldsymbol{\mathcal{D}}$.

5. **Either:**

(**a**) Show that there exists an infinite sequence $\mathbf{a}_0, \mathbf{a}_1, \ldots$ of degrees $\leq \mathbf{0}'$ such that for each $i \neq j$ we have $\mathbf{a}_i \mid \mathbf{a}_j$ (that is, $\mathbf{a}_i$ is incomparable with $\mathbf{a}_j$).

Or:

(**b**) Outline briefly a proof of the *Friedberg-Muchnik Theorem:*

There exists a pair $\mathbf{a}, \mathbf{b}$ *of incomparable computably enumerable Turing degrees .*

6. Write an essay, covering approximately *two to three pages*, describing the background to, and consequences of, Alan Turing's discovery of the existence of a Universal Turing Machine.

Your answer should contain enough mathematical content to show a good grasp of the notions and results involved, and enough discussion of these to show an understanding of the broader context.

END