MATH-312301

This question paper consists of 2 printed pages, each of which is identified by the Only approved basic scientific reference MATH-3123

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Examination for the Module MATH-3123
(January 2006)

## Set theory

Time allowed: 3 hours
Answer not more than four questions. All questions carry equal marks.

1. (i) State the axioms of pairing and replacement.
(ii) Show how the axiom of pairing can be derived from the other axioms of ZF set theory.
(iii) Determine, with reasons, which of the following abstraction terms can be proved in ZF to represent a set (where $a$ and $b$ are given to be sets):
(a) $\{x: a \in x \wedge x \subseteq b\}$,
(b) $\{x:(\exists y) x=\{y\}\}$,
(c) $\{x: x \in x\}$,
(d) $\{x: x \notin x\}$.
2. (i) Give the usual (Kuratowski) definition of 'ordered pair $\langle x, y\rangle$, and prove that it satisfies the basic property

$$
\langle x, y\rangle=\langle z, t\rangle \rightarrow x=z \wedge y=t
$$

(ii) Find sets $x, y, z, t$ such that $\{\{x\}\} \cup y=\{\{z\}\} \cup t$, but such that $x=z$ and $y=t$ do not both hold, and deduce that we cannot define 'ordered pair' by $\langle x, y\rangle=\{\{x\}\} \cup y$.
(iii) Starting from Von Neumann's definition of 'ordinal', prove that any member of an ordinal is an ordinal, and that any non-empty set of ordinals has a least member. (You may assume that ordinals are linearly ordered by $\epsilon$.)
3. (i) Define the transitive closure $T C(x)$ of a set $x$, and prove in ZF that $T C(x)$ is a set. Which axioms are used in the proof?
(ii) Prove that the axiom of extensionality holds in any transitive set, and that the power set axiom holds in $V_{\alpha}$ if and only if $\alpha$ is a limit ordinal.
(iii) Prove that the axiom of foundation holds in $\bigcup_{\alpha \in O n} V_{\alpha}$, without assuming that it holds in the universe $V$.
4. (i) Give a construction of the set $\mathbb{Z}$ of integers from the set $\mathbb{N}$ of natural numbers, assuming standard properties of $\mathbb{N}$. You should explain what the members of $\mathbb{Z}$ are taken to be, give the definitions of + and $\times$, verify that + is well-defined, and show how to identify $\mathbb{N}$ as a subset of $\mathbb{Z}$.
(ii) Define left Dedekind cut of the set $\mathbb{Q}$ of rational numbers, and prove that the family of left Dedekind cuts of $\mathbb{Q}$ is linearly ordered by inclusion, and is order-complete.
5. (i) State clearly the principle of definition by transfinite induction.
(ii) By appealing to part (i) or otherwise, prove that any set $X$ can be well-ordered, assuming that there is a choice function for the family of all its non-empty subsets.
(iii) Show that the following sets all have the same cardinality:
(a) $\mathbb{R}$,
(b) any open interval $(a, b)$ in $\mathbb{R}$ (where $a<b$ ),
(c) the set $2^{\omega}$ of all functions from $\omega$ into 2 ,
(d) the set of all subsets of the set $\mathbb{Z}$ of integers,
(e) the set of countable subsets of $\omega_{1}$.
[The Schröder-Bernstein Theorem may be used, but should be clearly stated.]

## END

