MATH-303201

This question paper consists of 4 printed pages, each of which is identified by the reference MATH 303201

No calculators allowed

Examination for the Module MATH 3032
(January 2005)

GRAPH THEORY

Time allowed : 2 hours

Do not answer more than *FOUR* questions.
All questions carry equal marks.

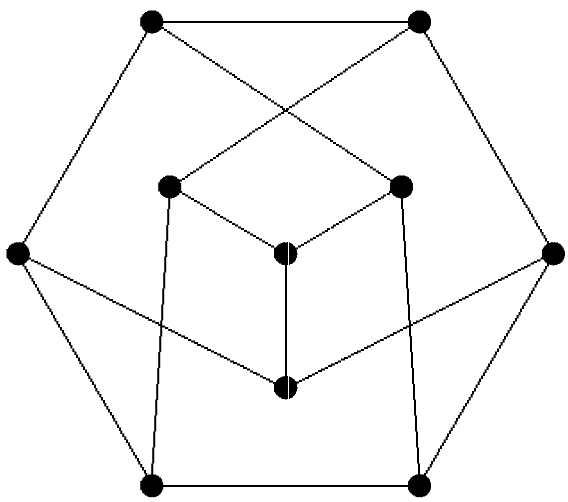
1. (**a**) For each of the following, *either* give an example, *or* explain why no such graph exists:

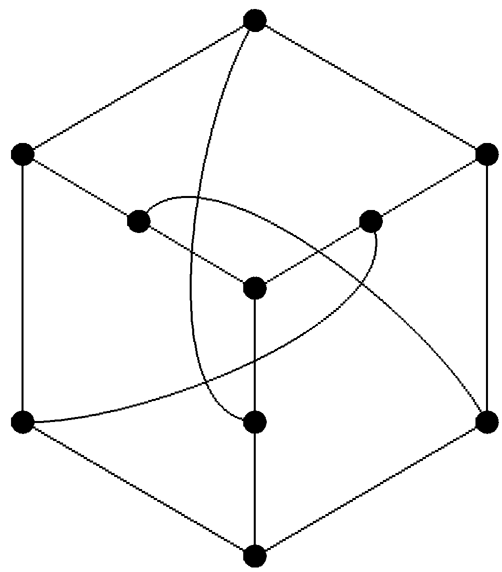
(i) a bipartite graph which is 4-regular,

(ii) a Platonic graph which is bipartite,

(iii) a cubic graph on seven vertices.

(**b**) Say, giving reasons, whether the two graphs below are isomorphic or not:





(**c**) Define: G is *connected*, H is a *component* of G, for G a graph.

Show that if G is a simple graph on ν vertices with ε edges and c components, then G satisfies

$$\varepsilon \leq \tfrac{1}{2}(\nu - c)(\nu - c + 1)\,.$$

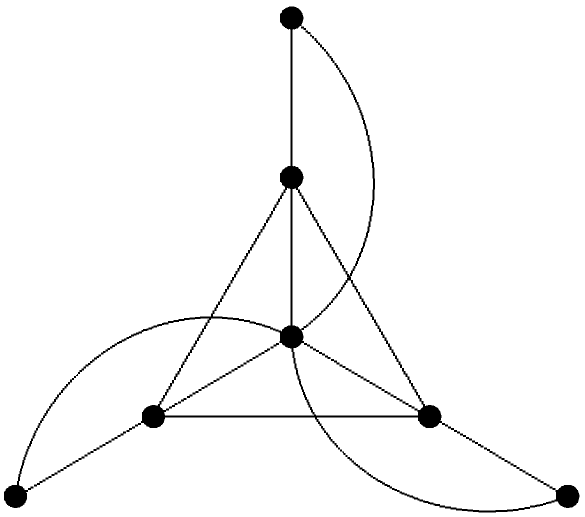
Say, giving reasons, whether there exists a disconnected simple graph with 7 vertices and 16 edges.

2. (**a**) Let G be a graph. Define: G is *Eulerian*, G is *Hamiltonian*.

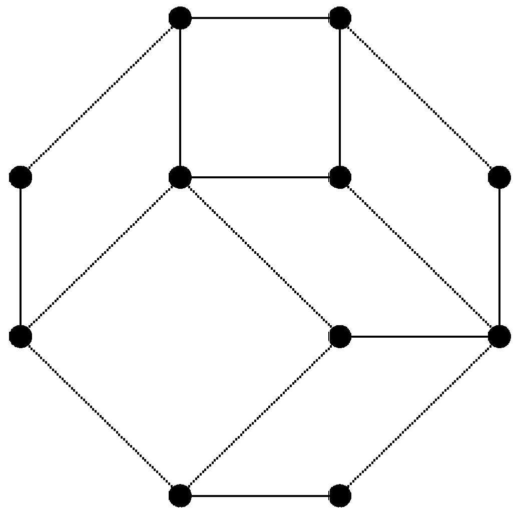
Show that a connected graph G is Eulerian if and only if the degree of each vertex of G is even.

[You may assume that every non-empty graph containing no vertices of degree < 2 contains a closed chain.]

(**b**) Say, giving reasons, which of the graphs G_1, G_2 below are Hamiltonian:



G_1

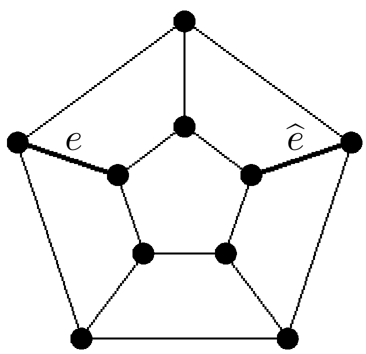


G_2

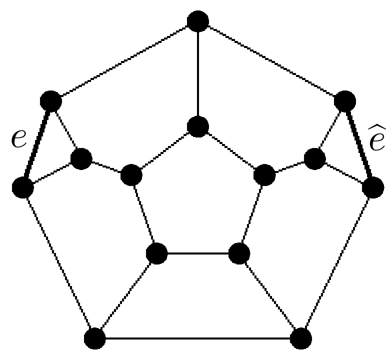
(**c**) Show that no Hamilton circuit in the graph G_1 below can contain both the edges e and $\widehat{e}$.

Hence show that no Hamilton circuit in the graph G_2 can contain both the edges e and $\widehat{e}$.

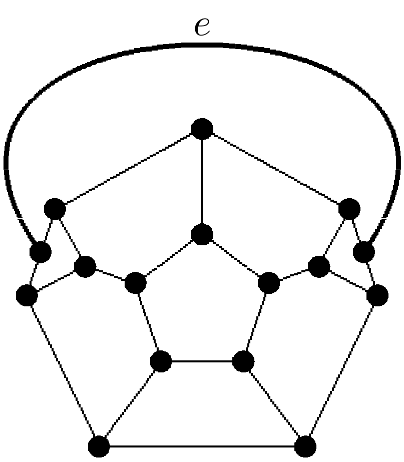
Deduce that every Hamilton circuit in the graph G_3 must contain the edge e.



G_1



G_2



G_3

continued ...

3. (**a**) Define: G is *planar*.

Prove *Euler's Formula* for a connected plane graph G with ν vertices, ε edges and φ faces:

$$\nu - \varepsilon + \varphi = 2\,.$$

(**b**) If G is a plane graph, define the *dual graph* G^* of G.

Deduce that $\nu(G^*) = \varphi(G)$, $\varepsilon(G^*) = \varepsilon(G)$ and $d_{G^*}(f^*) = d_G(f)$ for each face of G (where f^* is the vertex of G^* corresponding to f).

A plane graph is said to be *self-dual* if and only if it is isomorphic to its dual.

(i) Show that if G is self-dual, then $\varepsilon = 2\nu - 2$.

(ii) Find a self-dual plane graph on 6 vertices.

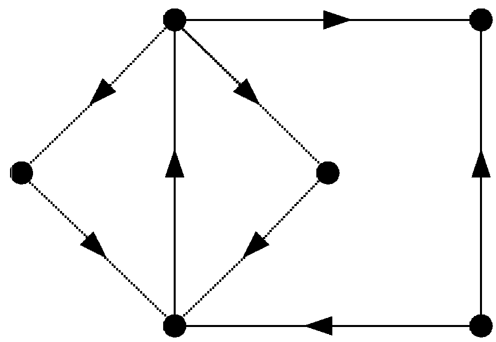
(**c**) Show that for $n = 5$, K_n (the complete graph on n vertices) is not planar, but that it can be embedded on the surface of a torus.

How would your answer change if $n = 6$? Justify your answer.

4. (**a**) Define the terms *strongly connected* and *dicomponent* for a digraph D.

Find the dicomponents for the digraph:

D_1



(**b**) Let $D_1, D_2, \ldots, D_m$ be the dicomponents of a digraph D.

The *condensation* $\widehat{D}$ of D is a directed graph with m vertices $w_1, w_2, \ldots, w_m$; there is an arc in $\widehat{D}$ with tail w_i and head w_j if and only if there is an arc in D with tail in D_i and head in D_j.

Sketch the condensation $\widehat{D}_1$ of the digraph D_1 in part (a).

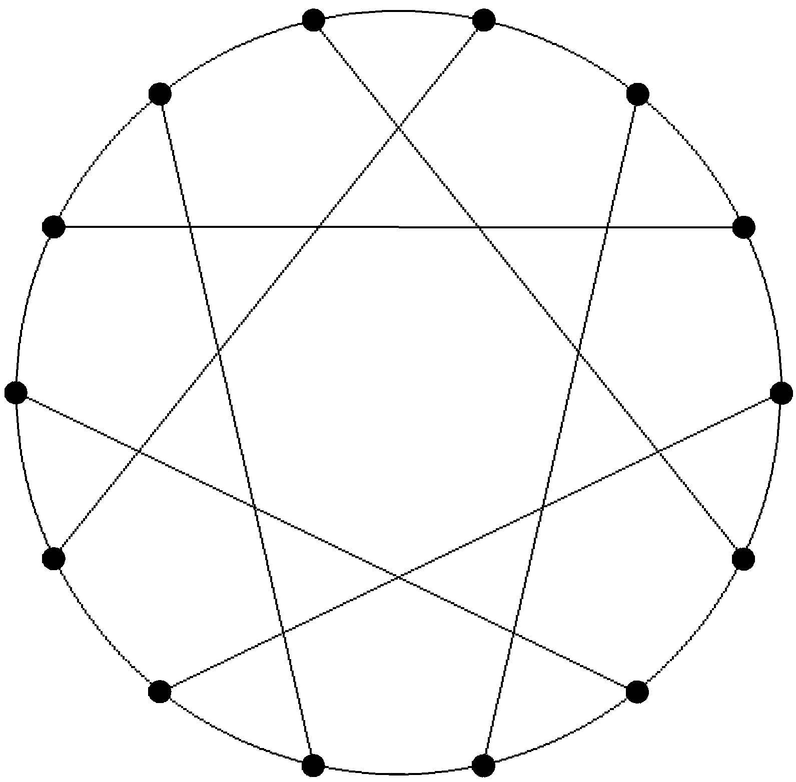
Show that, in general, the condensation $\widehat{D}$ of a digraph D contains no directed circuits.

(**c**) Show by induction on the number of vertices of D, or otherwise, that a loopless digraph D has an independent set S such that each vertex of D not in S is reachable from a vertex in S by a dipath of length ≤ 2.

Deduce that a tournament T contains a vertex v from which every other vertex is reachable by a dipath of length ≤ 2.

5. (**a**) Prove the *Five Colour Theorem.*
[You should carefully state any results you use from the theory of planar graphs.]

(**b**) Find an embedding of the Heawood graph (below) on the torus.



Deduce that the chromatic number $\chi(T)$ of a torus T is at least 7.
Use Heawood's inequality

$$\chi(S) \leq [\frac{1}{2}(7 + \sqrt{49 - 24n})]$$

for a surface S of Euler characteristic $n < 2$ to show that $\chi(T) = 7$.

END