**MATH275001**

This question paper consists of 4 printed pages, each of which is identified by the reference **MATH275001**.

Only approved basic scientific calculators may be used.



Examination for the Module MATH2750

(May-June 2006)

INTRODUCTION TO MARKOV PROCESSES

Time allowed: **2 hours**

Attempt no more than FOUR questions.

All questions carry equal marks.

**1.** (a) For a random process $(X_n)$ with state space $S = \{0, 1, 2, \dots\}$, give a mathematical definition of the *Markov property*. When is the Markov chain $(X_n)$ called *homogeneous* and what is meant by its *one-step transition probability* $p_{ij}$? State Chapman-Kolmogorov's equation to express the two-step transition probability $p_{ij}^{(2)}$.

(b) Stock traders pay close attention to the changes ("ticks") in a stock price, using the terms *down* (0), *even* (1), or *up* (2), according as the new price is lower, the same, or higher than the previous price. Traders have observed that, following an even tick, the next tick is even with probability $0.6$, while an up-tick and a down-tick have the same probability. A down-tick is followed by another down-tick with probability $0.3$, but never by an up-tick. Similarly, an up-tick is repeated with probability $0.3$, but is never followed by a down-tick. Explain briefly why the dynamics of the stock price ticks may be modelled by a Markov chain, and write down its transition probability matrix $P$.

(c) Suppose there are four cards labelled *a, b, c, d*, of which initially player A is given $a$ and $c$ and player B is given $b$ and $d$. A referee chooses at random a card from another deck of four cards *a, b, c, d*, and the player who has the same card gives it away to the other player. The game continues this way until one player gets all the four cards and wins.

If $X_n$ denotes the number of cards in possession of player A after $n$ steps, determine the transition probabilities of this Markov chain and sketch its transition graph.

Suppose that at some point of the game, player A has got three cards out of four.

(i) What is the probability that player A ultimately wins the game?

(ii) Obtain the expected remaining duration of the game (no matter who wins).

**2.** (a) Describe what is meant by the terms *transient* and *persistent* as applied to the states of a Markov chain. Give a criterion to discriminate between these two cases using the $n$-step return probabilities, and interpret it in terms of the *mean number* of returns to the state concerned.

(b) Explain briefly why any *finite* Markov chain (i.e., with finitely many states) must have at least one persistent state. Is the same true for any *infinite* Markov chain? (Explain briefly why if the answer is affirmative, or refer to a known example if the answer is negative.)

(c) Consider a Markov chain on the state space $S = \{1, 2, 3, 4, 5, 6, 7\}$ with transition probability matrix

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0.4 & 0.1 & 0.3 \\ 0.4 & 0 & 0 & 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0.2 & 0.3 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0.5 & 0 & 0 & 0 & 0.5 & 0 \end{pmatrix}$$

Sketch a transition graph of this chain. Determine which states are transient and which are persistent, and identify all closed irreducible subsets of states. Determine the periods of all states.

**3.** (a) Three of every four trucks on the road are followed by a car, while only one of every five cars is followed by a truck.

(i) Suppose you are driving a car. What is the probability that the second vehicle behind you is a truck?

(ii) What fraction of vehicles on the road are trucks?

(b) In a nuclear chain reaction, a free neutron with probability $p$ hits the nucleus of an atom and is replaced by two new free neutrons. Otherwise, with probability $q = 1 - p$, it remains inactive and is removed from the system. Assume that such a reaction may be modelled with a branching process. Denote by $X_n$ the number of free neutrons present in the system in the $n$-th generation, and suppose that initially there is one neutron, $X_0 = 1$.

(i) Let $G_n(s)$ be the probability generating function of $X_n$ $(n = 0, 1, 2, \dots)$. Explain why $G_0(s) = s$ for all $s$ and obtain an explicit expression for $G_1(s)$. Differentiating $G_1(s)$, check that the mean value $\mu_1 = \mathsf{E}(X_1)$ is given by $\mu_1 = 2p$.

(ii) Using the recursion

$$G_{n+1}(s) = G_1(G_n(s)), \qquad n = 0, 1, 2, \dots,$$

obtain $\mu_n = \mathsf{E}(X_n)$, the mean number of neutrons in the $n$-th generation.

(iii) For what values of $p$ will the reaction eventually cease with probability one? Discuss the answer in view of the behaviour of $\mu_n$ as $n \to \infty$ in part (ii).

(iv) Suppose specifically that $p = 0.8$, $q = 0.2$. What is the probability that the process will die out by the second generation? What is the probability $\eta$ of ultimate extinction?

**4.** (a) An ant crawls over a wire tetrahedron $A_1A_2A_3A_4$. From each vertex, the ant moves to one of the other three vertices chosen at random with probability $1/3$ each. If the ant starts at vertex $A_1$, find

(i) the probability to return to $A_1$ before visiting vertex $A_4$;

(ii) the mean number of visits to $A_4$ before the first return to $A_1$.

(b) During a given day, a retired mathematics professor, Dr Who, amuses himself with one of the following activities: (1) reading, (2) gardening, or (3) working on his book about space aliens. Suppose that he changes his activity from day to day according to a Markov chain with transition probability matrix

$$P = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/4 & 1/2 & 1/4 \\ 1/6 & 1/3 & 1/2 \end{pmatrix}$$

(i) Obtain the stationary distribution of the chain.

(ii) What proportion of days does Dr Who devote to each activity? Explain your answer carefully, referring to appropriate theory.

(iii) How many days will pass on average between two consecutive gardening activities?

**5.** (a) Suppose that major earthquakes (of magnitude $5.5$ or higher on the Richter scale) occur in a certain region of California in accordance with a Poisson process, at a rate of $0.15$ per year.

(i) What is the probability of at least two major earthquakes in one year?

(ii) Suppose there were two earthquakes during the past year. What is the probability that there will be no earthquakes during the following year?

(iii) Suppose that a major earthquake has a probability $0.015$ of damaging certain types of bridges. If a bridge of this type is constructed to last at least $60$ years, what is the probability that it will be undamaged by earthquakes during that period of time?

Obtain the required probabilities to $4$ significant figures.

(b) Consider a service station with a single serving facility. If the server is idle, a new customer arrives according to a Poisson process with rate $\lambda$ of four per hour. If the server is busy, a new arrival occurs at a reduced rate, specifically $\lambda/2$, and the new customer has to wait until the previous customer leaves. However, the waiting capacity is limited, so if there is already someone waiting for service, no more customers are accepted until the earlier service is completed. Each service time is exponentially distributed with mean $1/\mu = 5$ minutes.

Let $X_t$ be the number of customers in the system at time $t$, and denote

$$p_k(t) := \mathsf{P}(X_t = k), \qquad k = 0, 1, 2.$$

(i) By considering possible changes in the system over a small time interval $[t, t+h]$, show that

$$p_0(t+h) = p_0(t)(1 - \lambda h) + p_1(t)\mu h + o(h) \qquad (h \to 0).$$

Explain what is meant by the notation $o(h)$.
Obtain the analogous representations for $p_k(t+h)$, $k = 1, 2$.

(ii) Using part (i) derive the differential equations for the functions $p_k(t)$ $(k = 0, 1, 2)$. Check that the right-hand sides of these equations sum up to zero, and explain why.

(iii) Find the stationary distribution $\pi = (\pi_0, \pi_1, \pi_2)$. Explain what happens if the ratio $\rho = \lambda/\mu$ grows.

**END**