MATH275001

This question paper consists of 4 printed pages, each of which is identified by the reference **MATH275001**.

Only approved basic scientific calculators may be used.

Examination for the Module MATH2750

(May/June 2005)

INTRODUCTION TO MARKOV PROCESSES

Time allowed: **2 hours**

Attempt no more than FOUR questions.

All questions carry equal marks.

1. (a) For a process (X_n) with state space $S = \{0, 1, 2, \dots\}$, give a mathematical definition of the *Markov property*. When is the Markov chain (X_n) called *homogeneous* and what is meant by its *one-step transition probability* p_{ij}?

(b) A random walk on states $1, 2, 3, 4, 5, 6$ proceeds according to the following rule: "If you are at 3, throw a fair die to determine the next position; if you are at 6 go directly to 1; in any other case stay at i or go to $i + 1$ with equal probability". Explain why this is a Markov chain and determine its transition probability matrix P.

(c) In tennis, "deuce" is the state of a game with a particular (equal) score; to win the game from deuce, a player needs to gain a two-point advantage. Let 0 denote the state of deuce, and furthermore let "one-point advantage of player A" be represented by state 1; "game to player A" by state 2; "one-point advantage of player B" by state 3; "game to player B" by state 4. Assume that each point is played independently and the probability that player A wins the point equals p.

Suppose that a game has reached deuce.

(i) What is the probability that the game will ever end? Explain your answer using the method of difference equations.

(ii) Obtain the expected remaining duration of the game.

2. (a) Describe what is meant by the terms *transient* and *persistent* as applied to the states of a Markov chain. Give the criterion to discriminate between these two cases using the probabilities $p_{jj}^{(n)}$. Interpret this result in terms of the mean number of visits to the state concerned. Explain why any persistent state must be visited infinitely many times.

(b) Define what is meant by saying that "state i communicates with state j" (notation: $i \to j$). If state i is persistent and $i \to j$, is it true that $j \to i$? Is such a state j transient or persistent?

(c) Consider a Markov chain on the state space $S = \{1, 2, 3, 4, 5, 6, 7\}$ with a transition probability matrix

$$P = \begin{pmatrix} \frac{1}{8} & 0 & \frac{1}{4} & \frac{3}{8} & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{5} & 0 & \frac{2}{5} & 0 & \frac{2}{5} & 0 & 0 \\ 0 & \frac{3}{4} & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

Draw a transition graph of this chain. Determine which states are transient and which are persistent, and identify all closed irreducible subsets of states. Obtain the periods of all states.

3. Assume that certain bacteria reproduce according to a branching process. Specifically, suppose that a typical bacterium lives one second and then produces independently a random offspring of size $k = 0, 1, 2, \ldots$ with probabilities $g_k = (1-\alpha)\,\alpha^k$ (where $0 < \alpha < 1$). Suppose that initially there is a single bacterium, and let X_n be the number of bacteria living at time n (seconds).

(a) Show that the probability generating function of X_1 is given by

$$G(s) = \frac{1-\alpha}{1-\alpha s} \qquad (0 \le s \le 1).$$

Using this or otherwise, show that the mean offspring of one bacterium is given by

$$\mathsf{E}(X_1) = \frac{\alpha}{1-\alpha}\,.$$

(b) Let $G_n(s)$ be the probability generating function of X_n $(n = 0, 1, 2, \ldots)$. Expressing X_{n+1} as the sum of the offspring stemming from each of the bacteria present in the *first* generation, explain briefly the recurrence relation

$$G_{n+1}(s) = G(G_n(s)), \qquad n \ge 1.$$

(c) Using parts (a) and (b), obtain the mean number of bacteria at time n.

(d) Using part (b) or otherwise, obtain the probability that there are no bacteria left by time $n = 2$.

(e) Assuming specifically that $\alpha = 0.8$, find the probability of ultimate extinction. Obtain the same probability if the process starts with 4 bacteria.

4. (a) A particle performs a random walk on the vertices of a triangle ABC. At each step the particle remains where it is with probability $\frac{1}{2}$, and moves to each of the other two vertices with probability $\frac{1}{4}$. If the walk starts at vertex A, find

(i) the probability to return to A before passing the side AC in either direction;

(ii) the mean number of visits to B before the first return to A.

(b) In a big organisation, each employee has one of three possible job classifications and changes classifications (independently of the others) according to a Markov chain with transition probabilities

$$P = \begin{pmatrix} 0.45 & 0.46 & 0.09 \\ 0.04 & 0.67 & 0.29 \\ 0.01 & 0.50 & 0.49 \end{pmatrix}$$

(i) Obtain the stationary distribution of the chain.

(ii) In the long run, what percentage of employees are in each classification? Explain your answer carefully, referring to appropriate theory.

5. (a) Cars pass a point on the road according to a Poisson process at the rate of 3 per minute.

(i) What is the probability that no more than two cars pass by in a particular minute?

(ii) Suppose that 10 cars have passed by in 4 minutes. What is the probability that 5 cars passed by during the first two minutes?

(iii) What is the distribution of the time gap between two consecutive cars? Write down its probability density function. What is the expected amount of time until 15 cars pass by?

(b) Assume that calls arrive at a telephone exchange as a Poisson process with rate λ. The station has two trunk lines, and an incoming call is immediately connected if there is a free line, otherwise it is rejected and lost. The durations of conversations are independent and exponentially distributed with mean $1/\mu$.

Let $p_k(t)$ be the probability that at time t exactly k lines are busy $(k = 0, 1, 2)$.

(i) By considering possible transitions in the system on a small time interval $[t, t+h]$, show that

$$p_0(t+h) = p_0(t)(1 - \lambda h) + p_1(t)\mu h + o(h) \qquad (h \to 0).$$

Explain what is meant by the notation $o(h)$.
Obtain the analogous representations for $p_k(t+h)$, $k = 1, 2$.

(ii) Using part (i) derive the differential equations for the functions $p_k(t)$ $(k = 0, 1, 2)$.

(iii) Find the stationary distribution (π_k).

(iv) If the system is in equilibrium, what is the mean number of busy lines? Explain what happens if the traffic density $\rho = \lambda/\mu$ grows.

END