MATH275001

This question paper consists of 4 printed pages, each of which is identified by the reference **MATH275001**.

Only approved basic scientific calculators may be used.

Examination for the Module MATH2750

(May/June 2003)

INTRODUCTION TO MARKOV PROCESSES

Time allowed: **2 hours**

Attempt no more than FOUR questions.

All questions carry equal marks.

CONTINUED...

1. (a) For a stochastic process (X_n), $n = 0, 1, 2, \dots$ with state space $S = \{1, 2, 3, \dots\}$, give a mathematical definition which describes the *Markov property*.

When is a Markov chain called *homogeneous* and what is meant by its *one-step transition probability* p_{ij}?

(b) A player takes part in the following game: in each turn a fair die is rolled and the player gains £1 if the die shows 2, 4 or 6 and pays £1 if the die shows 1 or 5, whereas the outcome 3 has no effect. The player has initial capital of £10, and he has decided to quit playing once his fortune reaches £20, unless he has gone broke earlier.

(i) Explain how to model such a game with a Markov chain and describe its state space and transition probabilities.

(ii) Denote by Q_i the probability that the player will ultimately go broke having started with the initial capital of i pounds, $0 \le i \le 20$. Decomposing with respect to the outcome of the first turn, show that the function Q_i satisfies the difference equation

$$5Q_i = 3Q_{i+1} + 2Q_{i-1} \qquad (0 < i < 20).$$

(iii) Imposing the appropriate boundary conditions, solve the equation in part (ii) and compute the ruin probability Q_{10} to 4 significant figures.

(iv) How would the ruin probability change if the player did not have any stopping rule (except when being ruined)? Obtain this probability to 4 significant figures.

2. (a) Describe what is meant by the terms *transient*, *persistent*, *positive persistent* and *null persistent* as applied to the states of a Markov chain.

(b) Give an example of a Markov chain with an infinite state space, where one state is persistent and all other states are transient.

(c) Define the term *period of a state*. Find the period of state 1 for a Markov chain with transition probabilities

$$p_{12} = 0.5, \quad p_{13} = 0.5, \quad p_{21} = 1, \quad p_{32} = 1.$$

(d) Consider a Markov chain with state space $S = \{1, 2, 3, 4, 5, 6\}$ and transition matrix

$$P = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{2} & 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ \frac{3}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Draw a transition graph of this chain. Determine which states are transient and which are persistent, and identify all closed irreducible subsets of states.

3. Assume that dandelion spores reproduce according to a branching process. Specifically, a typical spore produces a random offspring of size $k = 0, 1, 2, \ldots$ with probabilities $\alpha^k(1-\alpha)$ $(0 < \alpha < 1)$, independently of the others and of the past history. Suppose that a single dandelion spore lands in an enclosed garden, and let Z_n be the number of spores in the nth generation.

(a) Show that the probability generating function of the offspring is given by

$$G(s) = \frac{1-\alpha}{1-\alpha s} \qquad (0 \leq s \leq 1).$$

Using this or otherwise, obtain the mean number of offspring.

(b) Let $G_n(s)$ denote the probability generating function of Z_n. Representing Z_n as the sum of the offspring stemming from Z_1 spores in the first generation, explain briefly the recursive relation

$$G_n(s) = G(G_{n-1}(s)) \qquad (n \geq 1).$$

(c) Using parts (a) and (b), show that the expected number of spores in the nth generation is given by

$$\mathsf{E}(Z_n) = \left(\frac{\alpha}{1-\alpha}\right)^n.$$

(d) Using part (b) or otherwise, obtain the probability that there are no spores in the second generation.

(e) Show that if $\alpha \leq 0.5$ then the population will ultimately become extinct with probability 1. Find the probability of ultimate extinction in the case $\alpha > 0.5$.

4. A family's income can be classed as Low $= 1$, Middle $= 2$ or High $= 3$. Suppose that from one generation to the next, families change their income group according to a Markov chain with the following transition matrix:

$$P = \begin{pmatrix} 0.6 & 0.3 & 0.1 \\ 0.2 & 0.7 & 0.1 \\ 0.1 & 0.3 & 0.6 \end{pmatrix}$$

(a) Using the method of difference equations, find the probability f_{13} that a family initially in a Low income group will ever enter a High income group.

(b) How many generations on average will pass until this happens?

(c) Obtain the stationary distribution of the chain.

(d) Does the equilibrium distribution exist for this chain? Explain your answer, referring to the appropriate theory.

Obtain the long-term proportions of the population in the three income groups.

Find the matrix limit $\lim_{n\to\infty} P^n$, where P^n is the nth power of the transition matrix P specified above.

5. (a) Describe a Poisson process (X_t) with rate λ. Write down the distribution of the random variable X_t and give an expression for the transition function $p_{ij}(t)$. Show that the expected value of $X_{s+t} - X_s$ equals λt.

(b) Assume that calls arrive at a telephone exchange as a Poisson process with rate λ, and the durations of conversations are independent and exponentially distributed with mean $1/\mu$. Suppose that there are always enough trunk lines (i.e., infinitely many), so there is no queuing and each incoming call is immediately connected. Let X_t be the number of calls in the system at time $t \geq 0$, and denote

$$p_n(t) = \mathsf{P}(X_t = n) \qquad (n = 0, 1, 2, \dots)$$

(i) By considering possible transitions in the system on a small time interval $[t, t+h]$, show that

$$p_0(t+h) = p_0(t)(1 - \lambda h) + p_1(t)\mu h + o(h).$$

Obtain the analogous representation for $p_n(t+h)$, $n \geq 1$. Passing to the limit as $h \to 0$ show that

$$\frac{dp_0(t)}{dt} = -\lambda p_0(t) + \mu p_1(t),$$

and derive the analogous differential equations for $p_n(t)$, $n \geq 1$.

(ii) Find the stationary distribution $\pi = (\pi_n)$ and verify that the probability that the system is empty is given by $\pi_0 = e^{-\rho}$, where $\rho = \lambda/\mu$.

END