MATH274001

This question paper consists of 12 printed pages, each of which is identified by the reference **MATH274001**.

Statistical tables are attached.
A formulae sheet is attached.
Only approved basic scientific calculators may be used.

Examination for the Module MATH2740
(May-June 2007)

Environmental Statistics

Time allowed: **2 hours**

Do not answer more than **four** questions.
All questions carry equal marks.

1. (a) Define, in terms of the mean number of points per unit area and the variance in the number of points per unit area: (i) a random point pattern; (ii) a clustered point pattern; (iii) a regular point pattern.

(b) (i) The number of clusters, M, in a given area follows a Poisson distribution with probability generating function:

$$G_M(s) = \exp[\lambda_1(s-1)], \quad (\lambda_1 > 0).$$

The number of points in the jth cluster is denoted by Y_j for $j = 1, 2, \ldots, M$ each following an independent Poisson distribution *shifted by a unit increase* (i.e. $Y_j - 1$ follows a Poisson distribution). It may be shown that each Y_j has a probability generating function:

$$G_Y(s) = s\exp[\lambda_2(s-1)], \quad (\lambda_2 > 0).$$

This distribution was first used by Thomas (1949) as a model for the distribution of plants of a given species in randomly placed quadrats. It is well suited to situations in which the parent as well as the offspring is included in the count for each cluster arising from an initial individual.

Briefly explain (in two sentences or less) why this might be an appropriate distribution under a parent-offspring model of plant locations.

(ii) The total number of points in the given area is calculated as $X = Y_1 + \cdots + Y_M$. Show that the probability generating function of X, $G_X(s)$ is given by:

$$G_X(s) = \exp\left[\lambda_1\{s\exp[\lambda_2(s-1)] - 1\}\right].$$

(iii) Use the probability generating function, $G_X(s)$ to show that:

$$\begin{aligned} \mathrm{P}(X=0) &= \exp\left[-\lambda_1\right], \\ \mathrm{P}(X=1) &= \lambda_1 \exp\left[-(\lambda_1+\lambda_2)\right]. \end{aligned}$$

(c) The table below gives the distribution of the number of *Acer pseudoplatanus* (sycamore) saplings in $N = 900$ quadrats each of size 25 square metres:

Number of saplings	0	1	2	3	4	5	6	7+	Total
Number of quadrats	$f_0 = 486$	$f_1 = 156$	130	63	31	16	13	5	900

Let the random variable X denote the number of saplings per quadrat. It has been suggested that the Thomas distribution with parameters λ_1 and λ_2 be used to model X.

(i) The Thomas distribution can be fitted by setting the observed proportion of quadrats containing zero and one saplings (f_0/N and f_1/N respectively) equal to their probabilities under the model. Show that this leads to parameter estimates $\widehat{\lambda}_1 = 0.6162$ and $\widehat{\lambda}_2 = 0.6521$ (to 4 decimal places).

(ii) The expected frequencies (to 2 decimal places) under the Thomas model are shown below.

Number of saplings	0	1	2	3	4	5	6	7+	Total
Number of quadrats	e_0	e_1	126.77	68.51	33.96	16.14	7.29	e_{7+}	900

Calculate the values to be inserted in place of e_0, e_1 and e_{7+}.

(iii) Perform a χ^2 goodness-of-fit test to assess the proposed model, and comment on the result.

2. (a) Explain how departure from a spatially random pattern affects the distance from a randomly chosen point to the nearest neighbouring plant assuming: (i) a clustered pattern; (ii) a regular pattern.

(b) Consider a Poisson forest of plants, with intensity λ plants per unit area. Let the random variable U denote the squared distance from a randomly selected point to its *second* nearest plant.

(i) Recall the probability density function of a random variable T following a gamma distribution with parameters r and $\alpha > 0$:

$$\mathrm{f}(t) = \frac{\alpha^r}{\Gamma(r)} t^{r-1} \exp(-\alpha t) \quad (t \geq 0).$$

The gamma function $\Gamma(r)$ has the form:

$$\Gamma(r) = \int_{z=0}^{\infty} z^{r-1} \exp(-z) \mathrm{d}z.$$

The random variable U follows a gamma distribution with parameters $r = 2$ and $\alpha = \lambda\pi$.

State the probability density function of U and show that:

$$\begin{aligned} \mathrm{E}(U) &= 2/(\lambda\pi), \\ \mathrm{Var}(U) &= 2/(\lambda\pi)^2. \end{aligned}$$

You may quote without proof that, for positive integer r, $\Gamma(r) = (r-1)!$.

(ii) Let the random variable $\overline{U} = (U_1 + \cdots + U_m)/m$ denote the mean of a set of m independent squared point to *second* nearest plant measurements from the Poisson forest. The random variable $\overline{U}$ follows a gamma distribution with parameters $r = 2m$ and $\alpha = \lambda\pi m$.

Use this result to derive the probability density function of $Y = 2m\lambda\pi\overline{U}$.

(iii) The random variable Y follows a gamma distribution. A gamma distribution with parameters $r = \nu/2$ and $\alpha = 1/2$ is a χ^2 distribution with ν degrees of freedom and thus has expectation ν.

Use this result to state $\mathrm{E}(Y)$.

Show how $\mathrm{E}(Y)$ can be derived from $\mathrm{E}(U) = 2/(\lambda\pi)$ using the definition $\mathrm{E}(Y) = 2m\lambda\pi\mathrm{E}(\overline{U})$.

(c) A quadrat sample reveals that there are, on average, 3.4 *Brassica rapa* (wild turnip) plants per 6.4 square metre quadrat. A set of $m = 20$ distances from a randomly selected point to the *second* nearest plant are taken, giving a mean of $\bar{u} = 0.80$ square metres. The test statistic $y = 2m\lambda\pi\bar{u}$ will follow a χ^2 distribution with $4m$ degrees of freedom under the null hypothesis that the plants are located at random.

Test the null hypothesis against an alternative that the locations are non-random. Briefly describe (in two sentences or less) any reservations you have about the test procedure.

3. (a) Briefly explain what is meant by the terms: (i) positive spatial autocorrelation; (ii) negative spatial autocorrelation.

(iii) State the values of the test statistics Z_{WW} and Z_{BW} that provide evidence against the null hypothesis of no spatial autocorrelation in favour of the alternative hypothesis of positive spatial autocorrelation at the 5% level.

(b) (i) Justify the use of the formula:

$$WW = \frac{1}{2}\sum_i \sum_j \delta_{ij}(1-x_i)(1-x_j),$$

for the number of white/white (i.e. WW) joins in a black/white map of spatial pattern, where:

$$x_i = \begin{cases} 1 & \text{if cell } i \text{ is } B \\ 0 & \text{if cell } i \text{ is } W \end{cases}$$

and δ_{ij} denotes the contiguity matrix defined as:

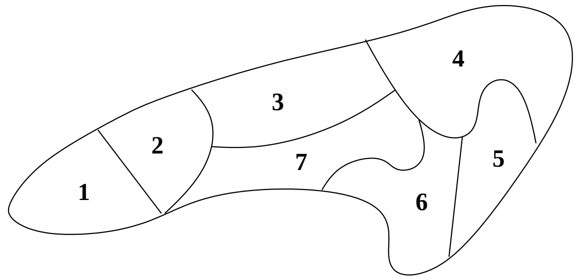
$$\delta_{ij} = \begin{cases} 1 & \text{if cells } i \text{ and } j \text{ are joined} \\ 0 & \text{otherwise} \end{cases}$$

with $\delta_{ii} = 0$ for all i.

(ii) Let L denote the total number of joins in the map. Let q denote the probability that a cell in the map is labelled W. Assume cells are labelled B and W independently of each other (i.e. free-sampling is assumed). Show that:

$$\mathrm{E}(WW) = Lq^2.$$

(c) A town is divided in to 7 regions as shown in the map below.



The proportion of households that recycle their garden waste in each region is show in the following table:

Region	1	2	3	4	5	6	7
Proportion recycling	0.62	0.53	0.71	0.41	0.45	0.32	0.28

(i) Nationally, the median proportion of households that recycle garden waste is 0.48. Denote regions in which the proportion recycling is above the national median as B and all other region as W. Regard two regions as joined if they meet at a boundary of non-zero length. Assess the evidence for the presence of positive spatial autocorrelation by calculating and interpreting the Z_{WW} test statistic.

(ii) Assuming Z_{WW} is normally distributed under the null hypothesis, what is the p-value associated with the test statistic you calculated in (i)? Do you have any reservations about the test procedure?

(iii) When the number of regions is small, rather than assuming normality, an alternative method to determine the distribution of WW under the null hypothesis of no spatial autocorrelation is Monte-Carlo testing. In this method the observed B and W regions are randomly re-assigned to the map and the join count statistic WW is re-evaluated. The procedure is repeated a fixed number of times to build up a distribution for WW. The table below shows the distribution of WW based on 99 random re-assignments to the map:

WW	1	2	3	4	5	Total
Number of occurrences	6	29	39	15	10	99

Large values of WW suggest positive spatial autocorrelation. Comment on whether the observed number of WW joins in part (i) appears large in relation to this distribution.

4. (a) (i) Explain what is meant by a saturated model.

(ii) State and interpret the formula that relates the AIC criterion to the (log) likelihood ratio statistic, Y^2.

(b) Let O denote the observed frequencies in a $r \times c$ contingency table, and E the expected frequencies under a log-linear model. Both the Pearson test statistic, X^2, and the likelihood ratio test statistic, Y^2 are members of a general class of *power divergence statistics*, given by:

$$D(\phi) = \frac{2}{\phi(\phi+1)} \sum [E + (O-E)][(1 + \tfrac{O-E}{E})^{\phi} - 1].$$

Assuming ϕ is a positive integer, and disregarding terms involving $(O-E)^k$ for $k \geq 3$, show that:

$$D(\phi) \approx X^2 = \sum \frac{(O-E)^2}{E},$$

provided that $|\frac{O-E}{E}| < 1$.

You may quote, without proof, the result:

$$(1+z)^{\phi} = \sum_{n=0}^{\infty} \binom{\phi}{n} z^n, \quad |z| < 1.$$

Hence state the approximate distribution of $D(\phi)$ under the null hypothesis that observed frequencies, O are a realisation from a model with ν degrees of freedom and expected frequencies, E.

(c) Howarth (1983) considered the relationship between the region, A, (measured in degrees latitude south) and the season, B, for the occurrence of cyclones near Antarctica between September 1973 and May 1975. The observed frequencies, O, are shown in the 3×4 table below:

Region / Season	Autumn	Winter	Spring	Summer	Total
40-49°S	370	452	273	422	1517
50-59°S	526	624	513	1059	2722
60-79°S	980	1200	995	1751	4926
Total	1876	2276	1781	3232	9165

The likelihood ratio statistic, Y^2, and the number of independent parameters, p, are reported below for various log-linear models fitted to the 3×4 table.

Model	AB	A/B	A	B	null
Y^2	0	71.34	617.00	2025.86	2571.52
p	12	6	3	4	1

(i) Calculate the AIC for each model, and determine which model is proposed by the forward-selection procedure.

(ii) The residuals, $O-E$ for the independence (A/B) model, are shown in the 3×4 table below:

Region / Season	Autumn	Winter	Spring	Summer	Total
40-49°S	`missing`	`missing`	-21.79	-112.96	0
50-59°S	`missing`	`missing`	-15.96	99.10	0
60-79°S	-28.31	-23.30	37.75	13.87	0
	0	0	0	0	

Calculate any values that are `missing`. What do these residuals tell you about the nature of any association between region, A, and the season, B, for the occurrence of cyclones near Antarctica?

5. A study was conducted on the association between education, business travel, and concern over the environmental impact of aviation. Data from the study were analysed using R. The variables were A, `plane`, (1=concerned about the impact of aviation on the environment, 2=not concerned), B, `travel`, (1=not travelled abroad in the last 12 months on business, 2=travelled abroad in the last 12 months on business) and C, `school`, (1=beyond elementary education, 2=elementary education only).

The aim of the analysis was to build a log-linear model which explained the significant factors and interactions present in the data. Consider the extract from an R session shown overleaf.

(a) Consider the output of the `summary(AB.model)` command. List the factors and interactions which are significant at the 5% level. Interpret any significant factors and interactions with reference to business travel and concern over the environmental impact of aviation.

(b) Consider the output of the `summary(ABC.model)` command. List the factors and interactions which are significant at the 5% level. Interpret any significant interactions with reference to education, business travel and concern over the environmental impact of aviation. Explain how and why your analysis differs from (a) above.

(c) Explain in detail the purpose of the `step()` command. Using hierarchy notation, write down the model selected by the `step()` command.

```
> options(contrasts=c("contr.sum","contr.poly"))

> AB.model = glm(count~plane*travel,family=poisson)
> summary(AB.model)

         Coefficients:
                          Estimate  Std. Error  z value  Pr(>|z|)
           (Intercept)     4.22230     0.04383   96.331   < 2e-16  ***
                plane1    -0.07478     0.04383   -1.706    0.0880  .
               travel1     0.29067     0.04383    6.632  3.32e-11  ***
         plane1:travel1   -0.09439     0.04383   -2.153    0.0313  *
                   ---
  Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.'  0.1 ' ' 1

> ABC.model = glm(count~plane*school*travel,family=poisson)
> summary(ABC.model)

         Coefficients:
                          Estimate  Std. Error  z value  Pr(>|z|)
           (Intercept)   4.119e+00   4.910e-02   83.889   < 2e-16  ***
                plane1  -9.346e-02   4.910e-02   -1.904     0.057  .
               school1   5.874e-02   4.910e-02    1.197     0.231
               travel1   3.059e-01   4.910e-02    6.231  4.64e-10  ***
         plane1:school1  3.587e-01   4.910e-02    7.307  2.73e-13  ***
         plane1:travel1  5.918e-03   4.910e-02    0.121     0.904
        school1:travel1 -3.000e-01   4.910e-02   -6.111  9.90e-10  ***
  plane1:school1:travel1 -3.502e-05  4.910e-02   -0.001     0.999
                   ---
  Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.'  0.1 ' ' 1

> forward.start = glm(count~plane+school+travel,family=poisson)
> try.models = list(upper=~plane*school*travel,lower=~plane+school+travel)
> step(forward.start,scope=try.models,direction="forward")

  Step:  AIC= 102.37
  count ~ plane + school + travel + plane:school
                             Df    Deviance      AIC
        + school:travel       1       0.015   59.692
         + plane:travel       1      40.044   99.721
                 <none>               44.690  102.368

  Step:  AIC= 59.69
  count ~ plane + school + travel + plane:school + school:travel
                             Df    Deviance      AIC
                 <none>                0.015   59.692
         + plane:travel       1    5.088e-07   61.677
```

Formulae Sheet

Poisson

$$P(X=x)=\frac{\exp(-\lambda)\lambda^x}{x!},\quad (x=0,1,\ldots).$$
$$E(X)=\text{Var}(X)=\lambda,\quad G_X(s)=\exp[\lambda(s-1)].$$

Logarithmic

$$P(X=x)=c\frac{q^x}{x},\quad (x=1,2,\ldots),\quad (0<q<1),\quad c=\frac{-1}{\ln(1-q)}.$$
$$E(X)=\frac{cq}{(1-q)},\quad G_X(s)=-c\ln(1-qs).$$

Neyman Type A

$$E(X)=\lambda_1\lambda_2,\quad \text{Var}(X)=\lambda_1\lambda_2(1+\lambda_2),\quad G_X(s)=\exp\left[\lambda_1\{\exp[\lambda_2(s-1)]-1\}\right].$$

Negative Binomial

$$P(X=x)=\binom{r+x-1}{x}p^r(1-p)^x,\quad (x=0,1,\ldots),\quad (0<p<1).$$
$$E(X)=\frac{r(1-p)}{p},\quad \text{Var}(X)=\frac{r(1-p)}{p^2},\quad G_X(s)=\left(\frac{p}{1-qs}\right)^r,\quad (q=1-p).$$

Gamma

$$f(x)=\frac{\alpha^r}{\Gamma(r)}x^{r-1}\exp(-\alpha x),\quad (x\ge 0),\quad (r,\alpha>0).$$
$$E(X)=\frac{r}{\alpha},\quad \text{Var}(X)=\frac{r}{\alpha^2}.$$

Weibull

$$f(r)=2\lambda\pi r\exp(-\lambda\pi r^2),\quad (r\ge 0),\quad (\lambda>0).$$
$$E(R)=\frac{1}{2\sqrt{\lambda}},\quad \text{Var}(R)=\frac{4-\pi}{4\lambda\pi}.$$

Testing for spatial randomness

$$Z=\frac{(n-1)s^2}{\bar{x}}\sim\chi^2_{n-1}.$$
$$E(\bar{r})=E(R)=\frac{1}{2\sqrt{\lambda}}.$$
$$\text{Var}(\bar{r})=\frac{\text{Var}(R)}{m}=\frac{4-\pi}{4\lambda\pi m}.$$
$$S=2m\widehat{\lambda}\pi\bar{u}\sim\chi^2_{2m}.$$
$$H=\frac{\sum r_{1i}^2}{\sum r_{2i}^2}\sim F_{2m,2m}.$$

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Spatial autocorrelation

$$L = \frac{1}{2}\sum_i L_i \text{ where } L_i \text{ is the number of cells joined to cell } i, \quad K = \frac{1}{2}\sum_i L_i(L_i - 1).$$

Free-sampling

$$\mathrm{E}(BB) = Lp^2, \quad \mathrm{E}(BW) = 2Lpq, \quad \mathrm{E}(WW) = Lq^2$$

$$\begin{aligned}
\mathrm{Var}(BB) &= Lp^2 + 2Kp^3 - (L + 2K)p^4. \\
\mathrm{Var}(BW) &= 2(L + K)pq - 4(L + 2K)p^2q^2. \\
\mathrm{Var}(WW) &= Lq^2 + 2Kq^3 - (L + 2K)q^4.
\end{aligned}$$

Non-free sampling

$$\begin{aligned}
\mathrm{E}(BB) &= L\frac{n_1(n_1-1)}{n(n-1)}. \\
\mathrm{E}(BW) &= 2L\frac{n_1n_2}{n(n-1)}. \\
\mathrm{E}(WW) &= L\frac{n_2(n_2-1)}{n(n-1)}.
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(BB) &= L\frac{n_1(n_1-1)}{n(n-1)} + 2K\frac{n_1(n_1-1)(n_1-2)}{n(n-1)(n-2)} \\
&+ [L(L-1) - 2K]\,\frac{n_1(n_1-1)(n_1-2)(n_1-3)}{n(n-1)(n-2)(n-3)} - \left[L\frac{n_1(n_1-1)}{n(n-1)}\right]^2.
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(BW) &= \frac{2(L+K)n_1n_2}{n(n-1)} + 4\,[L(L-1) - 2K]\,\frac{n_1(n_1-1)n_2(n_2-1)}{n(n-1)(n-2)(n-3)} \\
&- 4\left[\frac{Ln_1n_2}{n(n-1)}\right]^2.
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(WW) &= L\frac{n_2(n_2-1)}{n(n-1)} + 2K\frac{n_2(n_2-1)(n_2-2)}{n(n-1)(n-2)} \\
&+ [L(L-1) - 2K]\,\frac{n_2(n_2-1)(n_2-2)(n_2-3)}{n(n-1)(n-2)(n-3)} - \left[L\frac{n_2(n_2-1)}{n(n-1)}\right]^2.
\end{aligned}$$

Categorical data

For a 2×2 contingency table: $Y^2 = 2\sum_{ij} f_{ij}\ln(f_{ij}/e_{ij})$.

For the A/B model: $e_{ij} = \dfrac{f_{i0}f_{0j}}{f_{00}}$.

For the A model: $e_{ij} = \dfrac{f_{i0}}{2}$.

For the B model: $e_{ij} = \dfrac{f_{0j}}{2}$.

Normal Distribution Function Tables

The first table gives

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2} dt$$

and this corresponds to the shaded area in the figure to the right. $\Phi(x)$ is the probability that a random variable, normally distributed with zero mean amd unit variance, will be less than or equal to x. When $x < 0$ use $\Phi(x) = 1 - \Phi(-x)$, as the normal distribution with mean zero is symmetric about zero.

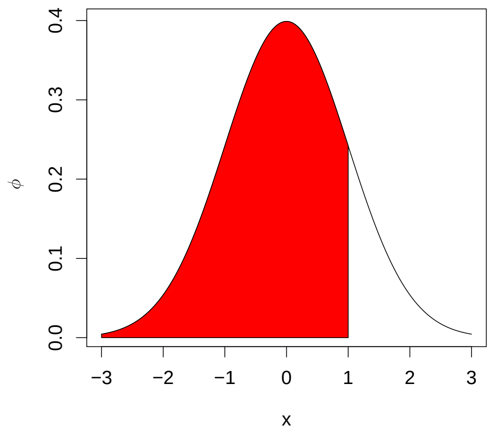

Table 1

x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$
0.00	0.5000	**0.50**	0.6915	**1.00**	0.8413	**1.50**	0.9332	**2.00**	0.9772	**2.50**	0.9938
0.05	0.5199	**0.55**	0.7088	**1.05**	0.8531	**1.55**	0.9394	**2.05**	0.9798	**2.55**	0.9946
0.10	0.5398	**0.60**	0.7257	**1.10**	0.8643	**1.60**	0.9452	**2.10**	0.9821	**2.60**	0.9953
0.15	0.5596	**0.65**	0.7422	**1.15**	0.8749	**1.65**	0.9505	**2.15**	0.9842	**2.65**	0.9960
0.20	0.5793	**0.70**	0.7580	**1.20**	0.8849	**1.70**	0.9554	**2.20**	0.9861	**2.70**	0.9965
0.25	0.5987	**0.75**	0.7734	**1.25**	0.8944	**1.75**	0.9599	**2.25**	0.9878	**2.75**	0.9970
0.30	0.6179	**0.80**	0.7881	**1.30**	0.9032	**1.80**	0.9641	**2.30**	0.9893	**2.80**	0.9974
0.35	0.6368	**0.85**	0.8023	**1.35**	0.9115	**1.85**	0.9678	**2.35**	0.9906	**2.85**	0.9978
0.40	0.6554	**0.90**	0.8159	**1.40**	0.9192	**1.90**	0.9713	**2.40**	0.9918	**2.90**	0.9981
0.45	0.6736	**0.95**	0.8289	**1.45**	0.9265	**1.95**	0.9744	**2.45**	0.9929	**2.95**	0.9984
0.50	0.6915	**1.00**	0.8413	**1.50**	0.9332	**2.00**	0.9772	**2.50**	0.9938	**3.00**	0.9987

The inverse function $\Phi^{-1}(p)$ is tabulated below for various values of p.

Table 2

p	**0.900**	**0.950**	**0.975**	**0.990**	**0.995**	**0.999**	**0.9995**
$\Phi^{-1}(p)$	1.2816	1.6449	1.9600	2.3263	2.5758	3.0902	3.2905

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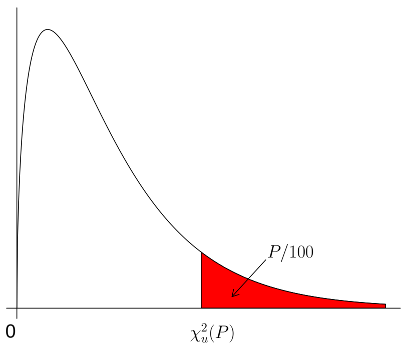
Percentage Points of the χ^2-Distribution

This table gives the percentage points $\chi^2_\nu(P)$ for various values of P and degrees of freedom ν, as indicated by the figure to the right.

If X is a variable distributed as χ^2 with ν degrees of freedom, $P/100$ is the probability that $X \geq \chi^2_\nu(P)$.

For $\nu > 100$, $\sqrt{2X}$ is approximately normally distributed with mean $\sqrt{2\nu - 1}$ and unit variance.



	Percentage points P					
ν	**99**	**97.5**	**95**	**5**	**2.5**	**1**
1	0.000	0.001	0.004	3.841	5.024	6.635
2	0.020	0.051	0.103	5.992	7.378	9.210
3	0.115	0.216	0.352	7.815	9.348	11.345
4	0.297	0.484	0.711	9.488	11.143	13.277
5	0.554	0.831	1.145	11.070	12.833	15.086
6	0.872	1.237	1.635	12.592	14.449	16.812
7	1.239	1.690	2.167	14.067	16.013	18.475
8	1.646	2.180	2.733	15.507	17.535	20.090
9	2.088	2.700	3.325	16.919	19.023	21.666
10	2.558	3.247	3.940	18.307	20.483	23.209
11	3.053	3.816	4.575	19.675	21.920	24.725
12	3.571	4.404	5.226	21.026	23.337	26.217
13	4.107	5.009	5.892	22.362	24.736	27.688
14	4.660	5.629	6.571	23.685	26.119	29.141
15	5.229	6.262	7.261	24.996	27.488	30.578
16	5.812	6.908	7.962	26.296	28.845	32.000
17	6.408	7.564	8.672	27.587	30.191	33.409
18	7.015	8.231	9.390	28.869	31.526	34.805
19	7.633	8.907	10.117	30.144	32.852	36.191
20	8.260	9.591	10.851	31.410	34.170	37.566
25	11.524	13.120	14.611	37.652	40.646	44.314
30	14.953	16.791	18.493	43.773	46.979	50.892
40	22.164	24.433	26.509	55.758	59.342	63.691
50	29.707	32.357	34.764	67.505	71.420	76.154
80	53.540	57.153	60.391	101.879	106.629	112.329

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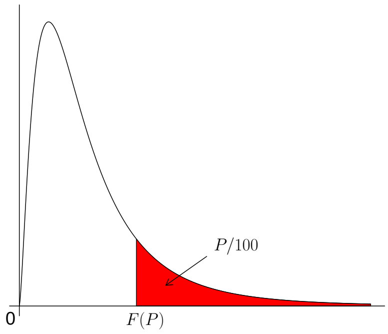
2.5 Percent Points of the *F*-Distribution

This table gives the percentage points $F_{\nu_1,\nu_2}(P)$ for $P = 0.025$ and degrees of freedom ν_1, ν_2, as indicated by the figure to the right.

The lower percentage points, that is the values $F'_{\nu_1,\nu_2}(P)$ such that the probability that $F \leq F'_{\nu_1,\nu_2}(P)$ is equal to $P/100$, may be found using the formula

$$F'_{\nu_1,\nu_2}(P) = 1/F_{\nu_1,\nu_2}(P)$$



ν_2 \ ν_1	1	2	3	4	5	6	12	24	∞
2	38.506	39.000	39.165	39.248	39.298	39.331	39.415	39.456	39.498
3	17.443	16.044	15.439	15.101	14.885	14.735	14.337	14.124	13.902
4	12.218	10.649	9.979	9.605	9.364	9.197	8.751	8.511	8.257
5	10.007	8.434	7.764	7.388	7.146	6.978	6.525	6.278	6.015
6	8.813	7.260	6.599	6.227	5.988	5.820	5.366	5.117	4.849
7	8.073	6.542	5.890	5.523	5.285	5.119	4.666	4.415	4.142
8	7.571	6.059	5.416	5.053	4.817	4.652	4.200	3.947	3.670
9	7.209	5.715	5.078	4.718	4.484	4.320	3.868	3.614	3.333
10	6.937	5.456	4.826	4.468	4.236	4.072	3.621	3.365	3.080
11	6.724	5.256	4.630	4.275	4.044	3.881	3.430	3.173	2.883
12	6.554	5.096	4.474	4.121	3.891	3.728	3.277	3.019	2.725
13	6.414	4.965	4.347	3.996	3.767	3.604	3.153	2.893	2.595
14	6.298	4.857	4.242	3.892	3.663	3.501	3.050	2.789	2.487
15	6.200	4.765	4.153	3.804	3.576	3.415	2.963	2.701	2.395
16	6.115	4.687	4.077	3.729	3.502	3.341	2.889	2.625	2.316
17	6.042	4.619	4.011	3.665	3.438	3.277	2.825	2.560	2.247
18	5.978	4.560	3.954	3.608	3.382	3.221	2.769	2.503	2.187
19	5.922	4.508	3.903	3.559	3.333	3.172	2.720	2.452	2.133
20	5.871	4.461	3.859	3.515	3.289	3.128	2.676	2.408	2.085
25	5.686	4.291	3.694	3.353	3.129	2.969	2.515	2.242	1.906
30	5.568	4.182	3.589	3.250	3.026	2.867	2.412	2.136	1.787
40	5.424	4.051	3.463	3.126	2.904	2.744	2.288	2.007	1.637
50	5.340	3.975	3.390	3.054	2.833	2.674	2.216	1.931	1.545
100	5.179	3.828	3.250	2.917	2.696	2.537	2.077	1.784	1.347
∞	5.024	3.689	3.116	2.786	2.567	2.408	1.945	1.640	1.003

END