MATH274001

This question paper consists of 12 printed pages, each of which is identified by the reference **MATH274001**.

Statistical tables are attached.
A formulae sheet is attached.
Only approved basic scientific calculators may be used.

Examination for the Module MATH2740
(May-June 2006)

Environmental Statistics

Time allowed: **2 hours**

Do not answer more than **four** questions.
All questions carry equal marks.

1. (a) Define, in terms of the mean number of points per unit area and the variance in the number of points per unit area: (i) a random point pattern; (ii) a clustered point pattern; (iii) a regular point pattern.

(b) (i) The number of clusters, M, in a given area follows a Poisson distribution with probability generating function:

$$G_M(s) = \exp[\lambda_1(s-1)], \quad (\lambda_1 > 0).$$

The number of points in the jth cluster is denoted by Y_j for $j = 1, 2, \ldots, M$ each following an independent Poisson distribution with probability generating function:

$$G_Y(s) = \exp[\lambda_2(s-1)], \quad (\lambda_2 > 0).$$

The total number of points in the given area is calculated as $X = Y_1 + \cdots + Y_M$.

Briefly explain, in terms of the number of points in a cluster, why the Poisson distribution is a theoretically inadequate model.

(ii) Show that the probability generating function of X, $G_X(s)$ is given by:

$$G_X(s) = \exp\left[\lambda_1\{\exp[\lambda_2(s-1)] - 1\}\right].$$

(iii) Use the probability generating function, $G_X(s)$ to show that:

$$\begin{aligned} \mathrm{P}(X=0) &= \exp\left[\lambda_1\{\exp(-\lambda_2) - 1\}\right], \\ \mathrm{E}(X) &= \lambda_1\lambda_2. \end{aligned}$$

(c) The table below gives the distribution of the number of *Antirrhinum majus* (snapdragon) plants in 300 quadrats each of size 4 square metres:

Number of plants per quadrat	0	1	2	3	4	5	6	7	8+	Total
Number of quadrats	165	62	24	12	13	8	5	11	0	300

Let the random variable X denote the number of plants per quadrat. It has been suggested that the Neyman Type A distribution with parameters λ_1 and λ_2 be used to model X.

(i) The Neyman Type A model can be fitted using the method of moments. Show that this leads to parameter estimates $\widehat{\lambda}_1 = 0.6092$ and $\widehat{\lambda}_2 = 1.8877$ (to 4 decimal places).

(ii) The expected frequencies (to 2 decimal places) under the Neyman Type A model are shown below.

Number of plants per quadrat	0	1	2	3	4	5	6
Number of quadrats	e_0	31.15	32.12	23.78	14.82	8.58	4.85

Number of plants per quadrat	7	8+	Total
Number of quadrats	2.70	e_{8+}	300

Calculate the values to be inserted in place of e_0 and e_{8+}.

(iii) Perform a χ^2 goodness-of-fit test to assess the proposed model and comment on the result.

2. (a) Explain how departure from a spatially random pattern affects the distance from a randomly chosen point to the nearest neighbouring plant assuming: (i) a clustered pattern; (ii) a regular pattern.

(b) Consider a Poisson forest of plants, with intensity λ plants per unit area. Let the random variable U denote the squared distance from a randomly selected point to the nearest plant.

(i) Consider the probability density function of U:

$$\mathrm{f}(u) = \lambda\pi \exp(-\lambda\pi u), \quad (u \geq 0).$$

Using integration by parts, or otherwise, show that:

$$\begin{aligned} \mathrm{E}(U) &= 1/(\lambda\pi), \\ \mathrm{Var}(U) &= 1/(\lambda\pi)^2. \end{aligned}$$

(ii) Let the random variable $\overline{U} = (U_1 + \cdots + U_m)/m$ denote the mean of a set of m independent squared point-plant measurements from the Poisson forest. Use the results from (i) to argue that:

$$\begin{aligned} \mathrm{E}(\overline{U}) &= 1/(\lambda\pi), \\ \mathrm{Var}(\overline{U}) &= 1/[m(\lambda\pi)^2]. \end{aligned}$$

(iii) Under the null hypothesis H_0 : spatially random pattern, the test statistic:

$$\frac{\overline{U} - 1/(\lambda\pi)}{\sqrt{1/[m(\lambda\pi)^2]}}$$

will approximate a standard normal distribution. Briefly explain why this is so, and describe the conditions under which the approximation is reasonable.

(c) A quadrat sample reveals that there are, on average, 2.5 *Eschscholzia californica* (Californian Poppy) plants per 3 square metre quadrat. A set of $m = 20$ squared point-plant measurements are taken, giving a mean of $\bar{u} = 0.60$ square metres. Use the result of (iii) above to test the null hypothesis that the plants are located at random against an alternative that the locations are non-random. Is the departure from randomness towards clustering or regularity? Do you have any reservations about the test?

3. (a) Briefly explain what is meant by the terms: (i) positive spatial autocorrelation; (ii) negative spatial autocorrelation; (iii) free-sampling.

(b) (i) A plant pathologist is studying the germination of *Triticum aestivum* (bread wheat). 100 seeds are sown in each of sixteen 10 square metre quadrats, arranged on a contiguous grid. Based on previous work, the pathologist believes that the proportion of wheat seeds that germinate in each quadrat, p, can be modelled by the probability density function:

$$\mathrm{f}(p) = \frac{p}{3} + a, \quad (0 < p < 1).$$

Derive the cumulative distribution function of, p, $\mathrm{F}(p)$. Hence determine the value of a and show that the median proportion of seeds that germinate is given by $m = 0.5414$ (to 4 decimal places).

(ii) Justify the use of the formula:

$$BW = \frac{1}{2} \sum_i \sum_j \delta_{ij} x_i (1 - x_j),$$

for the number of black/white (i.e. BW) joins in a black/white map of spatial pattern, where:

$$x_i = \begin{cases} 1 & \text{if cell } i \text{ is } B \\ 0 & \text{if cell } i \text{ is } W \end{cases}$$

QUESTION 3 CONTINUED...

and δ_{ij} denotes the contiguity matrix defined as:

$$\delta_{ij} = \begin{cases} 1 & \text{if cells } i \text{ and } j \text{ are joined} \\ 0 & \text{otherwise} \end{cases}$$

with $\delta_{ii} = 0$ for all i.

(c) (i) The pathologist wants to investigate the effect of spatial autocorrelation on the proportion of seeds that germinate. The proportions that germinate are summarised below:

0.0863	0.1367	0.2294	0.3190
0.1699	0.6748	0.8184	0.5610
0.9909	0.9726	0.2980	0.2010
0.9389	0.9293	0.7436	0.6304

By denoting quadrats in which the proportion of seeds that germinate is at or above m as B (and all other quadrats as W), show that the following black/white map is produced:

□	□	□	□
□	■	■	■
■	■	□	□
■	■	■	■

State whether free-or non-free sampling has been used.

(ii) Diagonal trends are of particular interest to the pathologist, and hence they wish to use the bishop's definition of contiguity. That is, *two quadrats are considered joined if and only if they meet at a corner but not at an edge.* Determine the number of BW joins using this definition of contiguity. It may be shown that (you do not need to check this) $L = 18$ and $K = 32$. Assess the evidence for the presence of spatial autocorrelation using BW joins. Comment on the outcome of the test, and the validity of any assumptions in the test procedure.

4. (a) Explain what is meant by: (i) a saturated model; (ii) the hierarchy principle.

(b) Consider the saturated model for a 2×2 contingency table with observed frequencies $\{f_{ij}\}$:

$$y_{ij} = \ln(f_{ij}) = \mu + \lambda_i^A + \lambda_j^B + \lambda_{ij}^{AB}, \quad (i = 1, 2), \quad (j = 1, 2),$$

subject to *cornered constraints*:

$$\lambda_2^A = \lambda_2^B = \lambda_{i2}^{AB} = \lambda_{2j}^{AB} = 0.$$

Show that:

$$\mu = y_{22}, \quad \lambda_1^A = y_{12} - y_{22},$$

$$\lambda_1^B = y_{21} - y_{22}, \quad \lambda_{11}^{AB} = y_{11} - y_{12} - y_{21} + y_{22}.$$

(c) An ecologist studies the attraction/repulsion between two species of butterfly *Pieris rapae* (cabbage white) and *Vanessa atalanta* (red admiral) by dividing a large area into 210 quadrats. The variables are A (1=cabbage white present, 2=cabbage white absent) and B (1=red admiral present, 2=red admiral absent).

	A_1	A_2
B_1	87	8
B_2	83	32

(i) Fit the saturated model to these data, obtaining parameter estimates $\widehat{\mu}$, $\widehat{\lambda}_1^A$, $\widehat{\lambda}_1^B$ and $\widehat{\lambda}_{11}^{AB}$.

(ii) Fit the independence model to these data and show that the expected frequencies, $\{e_{ij}\}$, have an odds ratio $\phi = 1$.

(iii) Using the likelihood ratio statistic, Y^2, test whether the independence model, A/B, is a good fit to the data. Interpret the result with reference to the attraction/repulsion of the two species of butterfly. Do you have any reservations about the experiment conducted?

5. A study was conducted on the association between gender, eating habits, and concern over factory farming. Data from the study were analysed using R. The variables were `gender` (1=male, 2=female), `meat` (1=eats meat, 2=does not eat meat) and `farm` (1=concerned about factory faming, 2=not concerned about factory farming).

The aim of the analysis was to build a log-linear model which explained the significant factors and interactions present in the data. Consider the following extract from an R session:

```
>  xtabs(count~meat+gender+farm,data=factory)

   , , farm = 1

         gender
   meat  1   2
      1  35 34
      2  22 51

   , , farm = 2

         gender
   meat  1   2
      1  43 40
      2  12 27

>  xtabs(count~meat+farm+gender,data=factory)

   , , gender = 1

         farm
   meat  1   2
      1  35 43
      2  22 12

   , , gender = 2

         farm
   meat  1   2
      1  34 40
      2  51 27

>  xtabs(count~farm+gender+meat,data=factory)

   , , meat = 1

         gender
   farm  1   2
      1  35 34
      2  43 40

   , , meat = 2

         gender
   farm  1   2
      1  22 51
      2  12 27
```


QUESTION 5 CONTINUED...

```
>  options(contrasts=c("contr.sum","contr.poly"))
>  saturated.model = glm(count~gender*meat*farm,family=poisson)
>  summary(saturated.model)

        Coefficients:
                          Estimate  Std. Error  z value  Pr(>|z|)
          (Intercept)    3.416925    0.067508    50.615   < 2e-16  ***
              gender1   -0.193801    0.067508    -2.871   0.00409  **
                meat1    0.216022    0.067508     3.200   0.00137  **
                farm1    0.109219    0.067508     1.618   0.10569
        gender1:meat1    0.219128    0.067508     3.246   0.00117  **
        gender1:farm1   -0.009148    0.067508    -0.136   0.89221
          meat1:farm1   -0.201312    0.067508    -2.982   0.00286  **
  gender1:meat1:farm1   -0.001685    0.067508    -0.025   0.98009
                  ---
  Signif.  codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.'  0.1 ' ' 1
```

(a) Let A denote the variable `gender`, B denote the variable `meat`, and C denote the variable `farm`. Estimate the odds-ratios for the six 2×2 tables generated by the `xtabs()` commands. Based on these estimated odds-ratios, use hierarchy notation to suggest a suitable model for the data. How many degrees of freedom are available to test the model you propose?

(b) Write down the expression you would insert in place of the ... in

```
fitted.model=glm(count~ ... ,family=poisson)
```

to fit the model you suggested in (a) using R.

(c) Consider the output of the `summary()` command. List the factors and interactions in the saturated model which are significant at the 5% level. Interpret any significant factors and interactions with reference to gender, eating habits, and concern over factory farming.

Formulae Sheet

Poisson

$$\mathrm{P}(X=x)=\frac{\exp(-\lambda)\lambda^x}{x!},\quad (x=0,1,\ldots).$$

$$\mathrm{E}(X)=\mathrm{Var}(X)=\lambda,\quad G_X(s)=\exp[\lambda(s-1)].$$

Logarithmic

$$\mathrm{P}(X=x)=c\frac{q^x}{x},\quad (x=1,2,\ldots),\quad (0<q<1),\quad c=\frac{-1}{\ln(1-q)}.$$

$$\mathrm{E}(X)=\frac{cq}{(1-q)},\quad G_X(s)=-c\ln(1-qs).$$

Neyman Type A

$$\mathrm{E}(X)=\lambda_1\lambda_2,\quad \mathrm{Var}(X)=\lambda_1\lambda_2(1+\lambda_2),\quad G_X(s)=\exp\left[\lambda_1\{\exp[\lambda_2(s-1)]-1\}\right].$$

Negative Binomial

$$\mathrm{P}(X=x)=\binom{r+x-1}{x}p^r(1-p)^x,\quad (x=0,1,\ldots),\quad (0<p<1).$$

$$\mathrm{E}(X)=\frac{r(1-p)}{p},\quad \mathrm{Var}(X)=\frac{r(1-p)}{p^2},\quad G_X(s)=\left(\frac{p}{1-qs}\right)^r,\quad (q=1-p).$$

Gamma

$$\mathrm{f}(x)=\frac{\alpha^r}{\Gamma(r)}x^{r-1}\exp(-\alpha x),\quad (x\ge 0),\quad (r,\alpha>0).$$

$$\mathrm{E}(X)=\frac{r}{\alpha},\quad \mathrm{Var}(X)=\frac{r}{\alpha^2}.$$

Weibull

$$\mathrm{f}(r)=2\lambda\pi r\exp(-\lambda\pi r^2),\quad (r\ge 0),\quad (\lambda>0).$$

$$\mathrm{E}(R)=\frac{1}{2\sqrt{\lambda}},\quad \mathrm{Var}(R)=\frac{4-\pi}{4\lambda\pi}.$$

Testing for spatial randomness

$$Z=\frac{(n-1)s^2}{\bar{x}}\sim\chi^2_{n-1}.$$

$$\mathrm{E}(\bar{r})=\mathrm{E}(R)=\frac{1}{2\sqrt{\lambda}}.$$

$$\mathrm{Var}(\bar{r})=\frac{\mathrm{Var}(R)}{m}=\frac{4-\pi}{4\lambda\pi m}.$$

$$S=2m\widehat{\lambda}\pi\bar{u}\sim\chi^2_{2m}.$$

$$H=\frac{\sum r_{1i}^2}{\sum r_{2i}^2}\sim \mathrm{F}_{2m,2m}.$$

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Spatial autocorrelation

$$L = \frac{1}{2}\sum_i L_i \text{ where } L_i \text{ is the number of cells joined to cell } i, \quad K = \frac{1}{2}\sum_i L_i(L_i - 1).$$

Free-sampling

$$\mathrm{E}(BB) = Lp^2, \quad \mathrm{E}(BW) = 2Lpq, \quad \mathrm{E}(WW) = Lq^2$$

$$\begin{aligned}
\mathrm{Var}(BB) &= Lp^2 + 2Kp^3 - (L+2K)p^4. \\
\mathrm{Var}(BW) &= 2(L+K)pq - 4(L+2K)p^2q^2. \\
\mathrm{Var}(WW) &= Lq^2 + 2Kq^3 - (L+2K)q^4.
\end{aligned}$$

Non-free sampling

$$\begin{aligned}
\mathrm{E}(BB) &= L\frac{n_1(n_1-1)}{n(n-1)}. \\
\mathrm{E}(BW) &= 2L\frac{n_1 n_2}{n(n-1)}. \\
\mathrm{E}(WW) &= L\frac{n_2(n_2-1)}{n(n-1)}.
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(BB) &= L\frac{n_1(n_1-1)}{n(n-1)} + 2K\frac{n_1(n_1-1)(n_1-2)}{n(n-1)(n-2)} \\
&+ [L(L-1) - 2K]\,\frac{n_1(n_1-1)(n_1-2)(n_1-3)}{n(n-1)(n-2)(n-3)} - \left[L\frac{n_1(n_1-1)}{n(n-1)}\right]^2.
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(BW) &= \frac{2(L+K)n_1n_2}{n(n-1)} + 4\,[L(L-1)-2K]\,\frac{n_1(n_1-1)n_2(n_2-1)}{n(n-1)(n-2)(n-3)} \\
&- 4\left[\frac{Ln_1n_2}{n(n-1)}\right]^2.
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(WW) &= L\frac{n_2(n_2-1)}{n(n-1)} + 2K\frac{n_2(n_2-1)(n_2-2)}{n(n-1)(n-2)} \\
&+ [L(L-1) - 2K]\,\frac{n_2(n_2-1)(n_2-2)(n_2-3)}{n(n-1)(n-2)(n-3)} - \left[L\frac{n_2(n_2-1)}{n(n-1)}\right]^2.
\end{aligned}$$

Categorical data

For a 2×2 contingency table: $Y^2 = 2\sum_{ij} f_{ij}\ln(f_{ij}/e_{ij})$.

For the A/B model: $e_{ij} = \dfrac{f_{i0}f_{0j}}{f_{00}}$.

For the A model: $e_{ij} = \dfrac{f_{i0}}{2}$.

For the B model: $e_{ij} = \dfrac{f_{0j}}{2}$.

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Normal Distribution Function Tables

The first table gives

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2} dt$$

and this corresponds to the shaded area in the figure to the right. $\Phi(x)$ is the probability that a random variable, normally distributed with zero mean amd unit variance, will be less than or equal to x. When $x < 0$ use $\Phi(x) = 1 - \Phi(-x)$, as the normal distribution with mean zero is symmetric about zero.

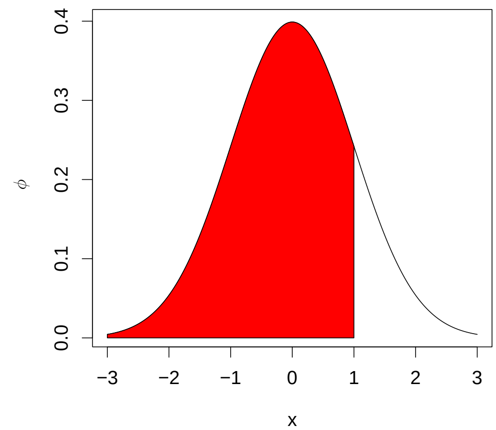

Table 1

x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$
0.00	0.5000	**0.50**	0.6915	**1.00**	0.8413	**1.50**	0.9332	**2.00**	0.9772	**2.50**	0.9938
0.05	0.5199	**0.55**	0.7088	**1.05**	0.8531	**1.55**	0.9394	**2.05**	0.9798	**2.55**	0.9946
0.10	0.5398	**0.60**	0.7257	**1.10**	0.8643	**1.60**	0.9452	**2.10**	0.9821	**2.60**	0.9953
0.15	0.5596	**0.65**	0.7422	**1.15**	0.8749	**1.65**	0.9505	**2.15**	0.9842	**2.65**	0.9960
0.20	0.5793	**0.70**	0.7580	**1.20**	0.8849	**1.70**	0.9554	**2.20**	0.9861	**2.70**	0.9965
0.25	0.5987	**0.75**	0.7734	**1.25**	0.8944	**1.75**	0.9599	**2.25**	0.9878	**2.75**	0.9970
0.30	0.6179	**0.80**	0.7881	**1.30**	0.9032	**1.80**	0.9641	**2.30**	0.9893	**2.80**	0.9974
0.35	0.6368	**0.85**	0.8023	**1.35**	0.9115	**1.85**	0.9678	**2.35**	0.9906	**2.85**	0.9978
0.40	0.6554	**0.90**	0.8159	**1.40**	0.9192	**1.90**	0.9713	**2.40**	0.9918	**2.90**	0.9981
0.45	0.6736	**0.95**	0.8289	**1.45**	0.9265	**1.95**	0.9744	**2.45**	0.9929	**2.95**	0.9984
0.50	0.6915	**1.00**	0.8413	**1.50**	0.9332	**2.00**	0.9772	**2.50**	0.9938	**3.00**	0.9987

The inverse function $\Phi^{-1}(p)$ is tabulated below for various values of p.

Table 2

p	**0.900**	**0.950**	**0.975**	**0.990**	**0.995**	**0.999**	**0.9995**
$\Phi^{-1}(p)$	1.2816	1.6449	1.9600	2.3263	2.5758	3.0902	3.2905

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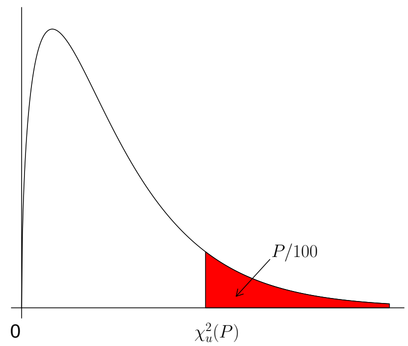
Percentage Points of the χ^2-Distribution

This table gives the percentage points $\chi^2_\nu(P)$ for various values of P and degrees of freedom ν, as indicated by the figure to the right.

If X is a variable distributed as χ^2 with ν degrees of freedom, $P/100$ is the probability that $X \geq \chi^2_\nu(P)$.

For $\nu > 100$, $\sqrt{2X}$ is approximately normally distributed with mean $\sqrt{2\nu - 1}$ and unit variance.



	Percentage points P					
ν	**99**	**97.5**	**95**	**5**	**2.5**	**1**
1	0.000	0.001	0.004	3.841	5.024	6.635
2	0.020	0.051	0.103	5.992	7.378	9.210
3	0.115	0.216	0.352	7.815	9.348	11.345
4	0.297	0.484	0.711	9.488	11.143	13.277
5	0.554	0.831	1.145	11.070	12.833	15.086
6	0.872	1.237	1.635	12.592	14.449	16.812
7	1.239	1.690	2.167	14.067	16.013	18.475
8	1.646	2.180	2.733	15.507	17.535	20.090
9	2.088	2.700	3.325	16.919	19.023	21.666
10	2.558	3.247	3.940	18.307	20.483	23.209
11	3.053	3.816	4.575	19.675	21.920	24.725
12	3.571	4.404	5.226	21.026	23.337	26.217
13	4.107	5.009	5.892	22.362	24.736	27.688
14	4.660	5.629	6.571	23.685	26.119	29.141
15	5.229	6.262	7.261	24.996	27.488	30.578
16	5.812	6.908	7.962	26.296	28.845	32.000
17	6.408	7.564	8.672	27.587	30.191	33.409
18	7.015	8.231	9.390	28.869	31.526	34.805
19	7.633	8.907	10.117	30.144	32.852	36.191
20	8.260	9.591	10.851	31.410	34.170	37.566
25	11.524	13.120	14.611	37.652	40.646	44.314
30	14.953	16.791	18.493	43.773	46.979	50.892
40	22.164	24.433	26.509	55.758	59.342	63.691
50	29.707	32.357	34.764	67.505	71.420	76.154
80	53.540	57.153	60.391	101.879	106.629	112.329

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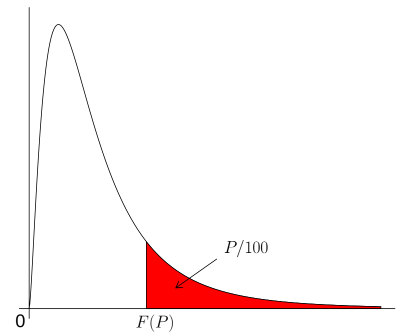
2.5 Percent Points of the F-Distribution

This table gives the percentage points $F_{\nu_1,\nu_2}(P)$ for $P = 0.025$ and degrees of freedom ν_1, ν_2, as indicated by the figure to the right.

The lower percentage points, that is the values $F'_{\nu_1,\nu_2}(P)$ such that the probability that $F \leq F'_{\nu_1,\nu_2}(P)$ is equal to $P/100$, may be found using the formula

$$F'_{\nu_1,\nu_2}(P) = 1/F_{\nu_1,\nu_2}(P)$$



	$\boldsymbol{\nu_1}$								
$\boldsymbol{\nu_2}$	**1**	**2**	**3**	**4**	**5**	**6**	**12**	**24**	$\boldsymbol{\infty}$
2	38.506	39.000	39.165	39.248	39.298	39.331	39.415	39.456	39.498
3	17.443	16.044	15.439	15.101	14.885	14.735	14.337	14.124	13.902
4	12.218	10.649	9.979	9.605	9.364	9.197	8.751	8.511	8.257
5	10.007	8.434	7.764	7.388	7.146	6.978	6.525	6.278	6.015
6	8.813	7.260	6.599	6.227	5.988	5.820	5.366	5.117	4.849
7	8.073	6.542	5.890	5.523	5.285	5.119	4.666	4.415	4.142
8	7.571	6.059	5.416	5.053	4.817	4.652	4.200	3.947	3.670
9	7.209	5.715	5.078	4.718	4.484	4.320	3.868	3.614	3.333
10	6.937	5.456	4.826	4.468	4.236	4.072	3.621	3.365	3.080
11	6.724	5.256	4.630	4.275	4.044	3.881	3.430	3.173	2.883
12	6.554	5.096	4.474	4.121	3.891	3.728	3.277	3.019	2.725
13	6.414	4.965	4.347	3.996	3.767	3.604	3.153	2.893	2.595
14	6.298	4.857	4.242	3.892	3.663	3.501	3.050	2.789	2.487
15	6.200	4.765	4.153	3.804	3.576	3.415	2.963	2.701	2.395
16	6.115	4.687	4.077	3.729	3.502	3.341	2.889	2.625	2.316
17	6.042	4.619	4.011	3.665	3.438	3.277	2.825	2.560	2.247
18	5.978	4.560	3.954	3.608	3.382	3.221	2.769	2.503	2.187
19	5.922	4.508	3.903	3.559	3.333	3.172	2.720	2.452	2.133
20	5.871	4.461	3.859	3.515	3.289	3.128	2.676	2.408	2.085
25	5.686	4.291	3.694	3.353	3.129	2.969	2.515	2.242	1.906
30	5.568	4.182	3.589	3.250	3.026	2.867	2.412	2.136	1.787
40	5.424	4.051	3.463	3.126	2.904	2.744	2.288	2.007	1.637
50	5.340	3.975	3.390	3.054	2.833	2.674	2.216	1.931	1.545
100	5.179	3.828	3.250	2.917	2.696	2.537	2.077	1.784	1.347
$\boldsymbol{\infty}$	5.024	3.689	3.116	2.786	2.567	2.408	1.945	1.640	1.003

END