MATH274001

This question paper consists of 12 printed pages, each of which is identified by the reference **MATH274001**.

Statistical tables are attached.
A formulae sheet is attached.
Only approved basic scientific calculators may be used.

UNIVERSITY OF LEEDS

Examination for the Module MATH2740
(May-June 2005)

Environmental Statistics

Time allowed: **2 hours**

Do not answer more than **four** questions.
All questions carry equal marks.

1. (a) Define, in terms of the mean number of points per unit area and the variance in the number of points per unit area: (i) a random point pattern; (ii) a clustered point pattern; (iii) a regular point pattern.

(b) (i) The random variable X follows a Bernoulli distribution with probability mass function

$$\begin{aligned} \mathrm{P}(X=0) &= 1-p, \\ \mathrm{P}(X=1) &= p, \qquad (0<p<1). \end{aligned}$$

Starting from the definition of the probability generating function,

$$G_X(s) = \mathrm{E}(s^X) = \sum_x s^x \mathrm{P}(X=x),$$

derive the probability generating function of X, $G_X(s)$.

(ii) Let $Y = X_1 + \ldots + X_n$, where $X_1, \ldots, X_n$ each follow independent Bernoulli distributions with common parameter p. Using the result of (i) show that the probability generating of Y, $G_Y(s)$, is given by:

$$G_Y(s) = [1 + p(s-1)]^n.$$

(iii) Using the result of (ii) show that:

$$\mathrm{E}(Y) = np.$$

(c) The table below gives the distribution of the number of *Prunus serotina* (black cherry) trees in 250 quadrats each of size 9 square metres:

Number of trees per quadrat	0	1	2	3	4	5	Total
Number of quadrats	20	68	99	49	11	3	250

Let the random variable Y denote the number of trees per quadrat. It has been suggested that the binomial distribution with parameters n and p be used to model Y.

(i) A plausible parameter estimation scheme is to equate n with the maximum observed number of trees per quadrat, and to estimate p by a method of moments (treating n as known). Show that this leads to estimates $\widehat{n} = 5$ and $\widehat{p} = 0.3776$.

(ii) Use the probability mass function:

$$\mathrm{P}(Y = y) = \binom{n}{y} p^y (1-p)^{n-y}, \quad (y = 0, 1, \ldots, n),$$

to determine the expected frequencies under a binomial model.

(iii) Perform a χ^2 goodness-of-fit test to assess the proposed model and comment on the result.

2. (a) Explain how departure from a spatially random pattern affects the distance from a randomly chosen point to the nearest neighbouring plant assuming: (i) a clustered pattern; (ii) a regular pattern.

(b) Consider a Poisson forest of plants, with intensity λ plants per unit area. Let the random variable $\overline{U}$ denote the mean of a set of m squared point-plant measurements.

(i) Consider the probability density function of $\overline{U}$:

$$\mathrm{f}(\bar{u}) = \frac{(\lambda \pi m)^m}{\Gamma(m)} \bar{u}^{m-1} \exp(-\lambda \pi m \bar{u}), \quad (\bar{u} \geq 0).$$

Using the substitution $y = \lambda \pi m \bar{u}$ show that:

$$\mathrm{E}(1/\overline{U}) = \frac{\lambda \pi m}{m-1}.$$

You may quote, without proof, the results:

$$\begin{aligned} \Gamma(k) &= \int_{z=0}^{\infty} z^{k-1} \exp(-z) \mathrm{d}z, \quad k > 0, \\ \Gamma(k) &= (k-1)!, \quad k = 1, 2, \ldots, \\ \mathrm{E}[\mathrm{g}(Z)] &= \int_{z=0}^{\infty} \mathrm{g}(z) \mathrm{f}(z) \mathrm{d}z, \end{aligned}$$

where $\mathrm{g}(z)$ is a real-valued function and $\mathrm{f}(z)$ is a probability density function, with $\mathrm{f}(z) = 0$ for $z < 0$.

(ii) Use the result from (i) to show that an unbiased estimator of λ in a Poisson forest of plants is given by:

$$\widetilde{\lambda} = \frac{m-1}{m\pi\bar{u}}.$$

(c) A census reveals that there are 12 *Brassica oleracea* (wild cabbage) plants in a 25 square metre study region. A set of $m = 15$ squared point-plant measurements are taken, giving a mean of $\bar{u} = 0.78$ square metres.

(i) By comparing $\widehat{\lambda}$ (the estimate of intensity based on the complete census) with $\widetilde{\lambda}$, determine whether any departure from spatial randomness is towards clustering or regularity.

(ii) Perform the Skellam-Moore test to determine whether there is significant evidence of a departure from spatial randomness in the locations of the *Brassica oleracea* plants.

3. (a) Briefly explain what is meant by the terms: (i) positive spatial autocorrelation; (ii) negative spatial autocorrelation.

(b) (i) Justify the use of the formula:

$$BB = \frac{1}{2}\sum_i \sum_j \delta_{ij} x_i x_j,$$

for the number of black/black (i.e. BB) joins in a black/white map of spatial pattern, where:

$$x_i = \begin{cases} 1 & \text{if cell } i \text{ is } B \\ 0 & \text{if cell } i \text{ is } W \end{cases}$$

and δ_{ij} denotes the contiguity matrix defined as:

$$\delta_{ij} = \begin{cases} 1 & \text{if cells } i \text{ and } j \text{ are joined} \\ 0 & \text{otherwise} \end{cases}$$

with $\delta_{ii} = 0$ for all i.

(ii) A set of observations $(X_1, Y_1), \ldots, (X_n, Y_n)$ are to be modelled by the regression equation:

$$Y_i = \beta X_i + \epsilon_i, \quad (i = 1, \ldots, n).$$

Show that the least-squares estimate of β is given by:

$$\widehat{\beta} = \frac{\sum_{i=1}^{n} X_i Y_i}{\sum_{i=1}^{n} X_i^2}.$$

QUESTION 3 CONTINUED...

(c) A plot of land used for growing wheat is divided into a grid of 3×3 equally sized quadrats. In each quadrat, the percentage of *Triticum aestivum* (common wheat) seeds that germinate, X_i, and the percentage of *Triticum monococcum* (einkorn wheat) seeds that germinate, Y_i, are recorded. The results, (X_i, Y_i), are summarised below:

(47 , 41)	(39 , 30)	(52 , 40)
(46 , 38)	(64 , 49)	(53 , 39)
(55 , 41)	(53 , 37)	(52 , 33)

$$\sum_{i=1}^{9} X_i = 461, \quad \sum_{i=1}^{9} X_i^2 = 23993, \quad \sum_{i=1}^{9} Y_i = 348, \quad \sum_{i=1}^{9} Y_i^2 = 13686, \quad \sum_{i=1}^{9} X_i Y_i = 18060.$$

(i) It has been suggested that the regression equation, $\widehat{Y}_i = \widehat{\beta} X_i$, be used to model the data. Calculate the least-squares estimator $\widehat{\beta}$. Hence fit the model to the data, recording the predicted values $\{\widehat{Y}_i\}$ and the residuals $\{\widehat{Y}_i - Y_i\}$. By denoting quadrats in which the residual is positive as B (and all other quadrats as W), show that the following black/white map is produced:

□	□	□
□	□	■
■	■	■

(ii) Assess the evidence for the presence of spatial autocorrelation in the residuals using BB joins. Comment on the outcome of the test, and the validity of any assumptions in the test procedure.

4. (a) Explain what is meant by: (i) a saturated model; (ii) a parsimonious model.

(b) (i) Consider a 2×2 contingency table in which the variables A and B are independent, and there are an equal number of observations in category 1 and category 2 of variable B. An appropriate model for the expected frequencies, $\{e_{ij}\}$, in this table is provided by the (multiplicative) A model:

$$e_{ij} = \eta \tau_i^A, \quad (i = 1, 2), \quad (j = 1, 2),$$

subject to the constraint:

$$\prod_i \tau_i^A = 1.$$

Show that:

$$\eta = (e_{11} e_{12} e_{21} e_{22})^{\frac{1}{4}}, \quad \tau_1^A = \left(\frac{e_{11} e_{12}}{e_{21} e_{22}} \right)^{\frac{1}{4}}.$$

(ii) Show that, for the model of (i), the expected frequencies, $\{e_{ij}\}$, have an odds ratio $\phi = 1$.

(c) The results of a survey into the relationship between gender and concern about declining fish stocks in the North Atlantic are summarised in the following table. The variables are A (1=male, 2=female) and B (1=concerned about declining fish stocks, 2=not concerned about declining fish stocks).

	A_1	A_2
B_1	89	44
B_2	76	53

(i) Show that the expected frequencies, $\{e_{ij}\}$, for the A model are:

$$e_{11} = 82.5, \quad e_{12} = 82.5, \quad e_{21} = 48.5, \quad e_{22} = 48.5.$$

(ii) Use these expected frequencies to obtain parameter estimates $\widehat{\eta}$ and $\widehat{\tau}_1^A$ for the A model.

(iii) Using the likelihood-ratio statistic, Y^2, test whether the A model is a good fit to the data. Interpret the result with reference to gender and concern over declining fish stocks.

(iv) The null model:

$$e_{ij} = \eta, \quad (i = 1, 2), \quad (j = 1, 2),$$

has a likelihood-ratio statistic $Y^2 = 19.71$ (correct to 2 decimal places) when fitted to these data. Use this, together with the Y^2 value calculated in (iii), to test whether removing the parameter τ_1^A significantly affects the explanation provided by the A model.

5. A study was conducted on the association between gender, marital status and support for house building on green belt land. Data from the study were analysed using R. The variables were `gender` (1=female, 2=male), `marital` (1=married, 2=not married) and `build` (1=supports house building on green belt land, 2=does not support house building on green belt land). Let A denote the variable `gender`, B denote the variable `marital`, and C denote the variable `build`.

The aim of the analysis was to build a log-linear model which explained the significant factors and interactions present in the data. Consider the following extract from an R session:

```
>   options(contrasts=c("contr.sum","contr.poly"))
>   ABC.model = glm(count~gender*marital*build,family=poisson)
>   summary(ABC.model)

             Coefficients:
                                Estimate  Std.  Error  z value  Pr(>|z|)
              (Intercept)   3.876570     0.052880   73.309   < 2e-16  ***
                  gender1  -0.213437     0.052880   -4.036  5.43e-05  ***
                 marital1  -0.265347     0.052880   -5.018  5.22e-07  ***
                   build1  -0.090318     0.052880   -1.708    0.0876  .
         gender1:marital1  -0.001613     0.052880   -0.031    0.9757
           gender1:build1  -0.133136     0.052880   -2.518    0.0118  *
          marital1:build1   0.121506     0.052880    2.298    0.0216  *
  gender1:marital1:build1   0.001613     0.052880    0.031    0.9757
                      ---
    Signif.  codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.'  0.1 ' ' 1

>   AIC(ABC.model)
[1] 61.7454
>   AB.AC.BC.model = glm(count~gender*marital+gender*build+marital*build,
    family=poisson)
>   AIC(AB.AC.BC.model)
[1] 59.74633
>   AB.AC.model = glm(count~gender*marital+gender*build,family=poisson)
>   AIC(AB.AC.model)
[1] 63.28473
>   AB.BC.model = glm(count~gender*marital+marital*build,family=poisson)
>   AIC(AB.BC.model)
[1] 64.62736
>   AC.BC.model = glm(count~gender*build+marital*build,family=poisson)
>   AIC(AC.BC.model)
[1] 57.74744
>   AB.C.model = glm(count~gender*marital+build,family=poisson)
>   AIC(AB.C.model)
[1] 68.28026
>   AC.B.model = glm(count~gender*build+marital,family=poisson)
>   AIC(AC.B.model)
[1] 61.40034
>   BC.A.model = glm(count~marital*build+gender,family=poisson)
>   AIC(BC.A.model)
[1] 62.74297
>   A.B.C.model = glm(count~gender+marital+build,family=poisson)
>   AIC(A.B.C.model)
[1] 66.39587
```


QUESTION 5 CONTINUED...

Use the extract from the R session to answer the following questions.

(a) Briefly explain the use of the `*` symbol in the `glm()` commands.

(b) Consider the output of the `summary()` command. List the factors and interactions in the saturated model which are significant at the 5% level. Interpret any significant factors and interactions with reference to gender, marital status and support for house building on green belt land.

(c) Use the output from the `AIC()` commands to carry out the process of forward selection on these data. Illustrate your answer by sketching the appropriate model tree. Mark on the tree the branches the forward selection process moves along. How many degrees of freedom are available to test the model proposed when the forward selection process terminates?

Formulae Sheet

Poisson

$$\mathrm{P}(X = x) = \frac{\exp(-\lambda)\lambda^x}{x!}, \quad (x = 0, 1, \ldots).$$
$$\mathrm{E}(X) = \mathrm{Var}(X) = \lambda, \quad G_X(s) = \exp[\lambda(s-1)].$$

Logarithmic

$$\mathrm{P}(X = x) = c\frac{q^x}{x}, \quad (x = 1, 2, \ldots), \quad (0 < q < 1), \quad c = \frac{-1}{\ln(1-q)}.$$
$$\mathrm{E}(X) = \frac{cq}{(1-q)}, \quad G_X(s) = -c\ln(1-qs).$$

Neyman Type A

$$\mathrm{E}(X) = \lambda_1\lambda_2, \quad \mathrm{Var}(X) = \lambda_1\lambda_2(1+\lambda_2), \quad G_X(s) = \exp\left[\lambda_1\{\exp[\lambda_2(s-1)] - 1\}\right].$$

Negative Binomial

$$\mathrm{P}(X = x) = \binom{r+x-1}{x} p^r(1-p)^x, \quad (x = 0, 1, \ldots), \quad (0 < p < 1).$$
$$\mathrm{E}(X) = \frac{r(1-p)}{p}, \quad \mathrm{Var}(X) = \frac{r(1-p)}{p^2}, \quad G_X(s) = \left(\frac{p}{1-qs}\right)^r, \quad (q = 1-p).$$

Gamma

$$\mathrm{f}(x) = \frac{\alpha^r}{\Gamma(r)} x^{r-1}\exp(-\alpha x), \quad (x \geq 0), \quad (r, \alpha > 0).$$
$$\mathrm{E}(X) = \frac{r}{\alpha}, \quad \mathrm{Var}(X) = \frac{r}{\alpha^2}.$$

Weibull

$$\mathrm{f}(r) = 2\lambda\pi r\exp(-\lambda\pi r^2), \quad (r \geq 0), \quad (\lambda > 0).$$
$$\mathrm{E}(R) = \frac{1}{2\sqrt{\lambda}}, \quad \mathrm{Var}(R) = \frac{4-\pi}{4\lambda\pi}.$$

Testing for spatial randomness

$$Z = \frac{(n-1)s^2}{\bar{x}} \sim \chi^2_{n-1}.$$
$$\mathrm{E}(\bar{r}) = \mathrm{E}(R) = \frac{1}{2\sqrt{\lambda}}.$$
$$\mathrm{Var}(\bar{r}) = \frac{\mathrm{Var}(R)}{m} = \frac{4-\pi}{4\lambda\pi m}.$$
$$S = 2m\widehat{\lambda}\pi\bar{u} \sim \chi^2_{2m}.$$
$$H = \frac{\sum r_{1i}^2}{\sum r_{2i}^2} \sim \mathrm{F}_{2m,2m}.$$

CONTINUED...

Spatial autocorrelation

$$L = \frac{1}{2}\sum_i L_i \text{ where } L_i \text{ is the number of cells joined to cell } i, \quad K = \frac{1}{2}\sum_i L_i(L_i - 1).$$

Free-sampling

$$\mathrm{E}(BB) = Lp^2, \quad \mathrm{E}(BW) = 2Lpq, \quad \mathrm{E}(WW) = Lq^2$$

$$\begin{aligned}
\mathrm{Var}(BB) &= Lp^2 + 2Kp^3 - (L + 2K)p^4. \\
\mathrm{Var}(BW) &= 2(L + K)pq - 4(L + 2K)p^2q^2. \\
\mathrm{Var}(WW) &= Lq^2 + 2Kq^3 - (L + 2K)q^4.
\end{aligned}$$

Non-free sampling

$$\begin{aligned}
\mathrm{E}(BB) &= L\frac{n_1(n_1 - 1)}{n(n-1)}. \\
\mathrm{E}(BW) &= 2L\frac{n_1 n_2}{n(n-1)}. \\
\mathrm{E}(WW) &= L\frac{n_2(n_2 - 1)}{n(n-1)}.
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(BB) &= L\frac{n_1(n_1-1)}{n(n-1)} + 2K\frac{n_1(n_1-1)(n_1-2)}{n(n-1)(n-2)} \\
&+ [L(L-1) - 2K]\frac{n_1(n_1-1)(n_1-2)(n_1-3)}{n(n-1)(n-2)(n-3)} - \left[L\frac{n_1(n_1-1)}{n(n-1)}\right]^2.
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(BW) &= \frac{2(L+K)n_1 n_2}{n(n-1)} + 4[L(L-1) - 2K]\frac{n_1(n_1-1)n_2(n_2-1)}{n(n-1)(n-2)(n-3)} \\
&- 4\left[\frac{Ln_1 n_2}{n(n-1)}\right]^2.
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(WW) &= L\frac{n_2(n_2-1)}{n(n-1)} + 2K\frac{n_2(n_2-1)(n_2-2)}{n(n-1)(n-2)} \\
&+ [L(L-1) - 2K]\frac{n_2(n_2-1)(n_2-2)(n_2-3)}{n(n-1)(n-2)(n-3)} - \left[L\frac{n_2(n_2-1)}{n(n-1)}\right]^2.
\end{aligned}$$

Categorical data

For a 2×2 contingency table: $Y^2 = 2\sum_{ij} f_{ij} \ln(f_{ij}/e_{ij})$.

For the A/B model: $e_{ij} = \dfrac{f_{i0}f_{0j}}{f_{00}}$.

For the A model: $e_{ij} = \dfrac{f_{i0}}{2}$.

For the B model: $e_{ij} = \dfrac{f_{0j}}{2}$.

Normal Distribution Function Tables

The first table gives

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2}\, dt$$

and this corresponds to the shaded area in the figure to the right. $\Phi(x)$ is the probability that a random variable, normally distributed with zero mean amd unit variance, will be less than or equal to x. When $x < 0$ use $\Phi(x) = 1 - \Phi(-x)$, as the normal distribution with mean zero is symmetric about zero.

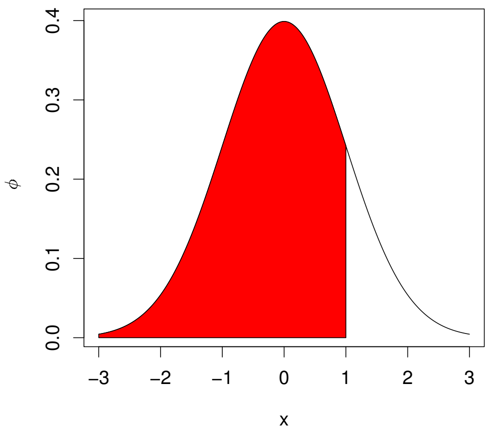


Table 1

x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$
0.00	0.5000	**0.50**	0.6915	**1.00**	0.8413	**1.50**	0.9332	**2.00**	0.9772	**2.50**	0.9938
0.05	0.5199	**0.55**	0.7088	**1.05**	0.8531	**1.55**	0.9394	**2.05**	0.9798	**2.55**	0.9946
0.10	0.5398	**0.60**	0.7257	**1.10**	0.8643	**1.60**	0.9452	**2.10**	0.9821	**2.60**	0.9953
0.15	0.5596	**0.65**	0.7422	**1.15**	0.8749	**1.65**	0.9505	**2.15**	0.9842	**2.65**	0.9960
0.20	0.5793	**0.70**	0.7580	**1.20**	0.8849	**1.70**	0.9554	**2.20**	0.9861	**2.70**	0.9965
0.25	0.5987	**0.75**	0.7734	**1.25**	0.8944	**1.75**	0.9599	**2.25**	0.9878	**2.75**	0.9970
0.30	0.6179	**0.80**	0.7881	**1.30**	0.9032	**1.80**	0.9641	**2.30**	0.9893	**2.80**	0.9974
0.35	0.6368	**0.85**	0.8023	**1.35**	0.9115	**1.85**	0.9678	**2.35**	0.9906	**2.85**	0.9978
0.40	0.6554	**0.90**	0.8159	**1.40**	0.9192	**1.90**	0.9713	**2.40**	0.9918	**2.90**	0.9981
0.45	0.6736	**0.95**	0.8289	**1.45**	0.9265	**1.95**	0.9744	**2.45**	0.9929	**2.95**	0.9984
0.50	0.6915	**1.00**	0.8413	**1.50**	0.9332	**2.00**	0.9772	**2.50**	0.9938	**3.00**	0.9987

The inverse function $\Phi^{-1}(p)$ is tabulated below for various values of p.

Table 2

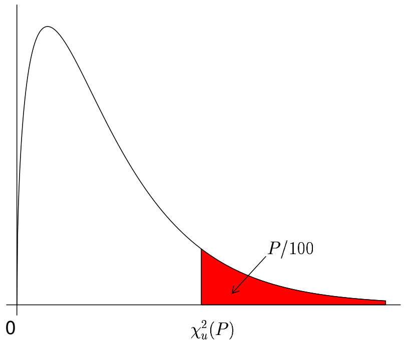
p	**0.900**	**0.950**	**0.975**	**0.990**	**0.995**	**0.999**	**0.9995**
$\Phi^{-1}(p)$	1.2816	1.6449	1.9600	2.3263	2.5758	3.0902	3.2905

Percentage Points of the χ^2-Distribution

This table gives the percentage points $\chi^2_\nu(P)$ for various values of P and degrees of freedom ν, as indicated by the figure to the right.

If X is a variable distributed as χ^2 with ν degrees of freedom, $P/100$ is the probability that $X \geq \chi^2_\nu(P)$.

For $\nu > 100$, $\sqrt{2X}$ is approximately normally distributed with mean $\sqrt{2\nu - 1}$ and unit variance.



ν	Percentage points P					
	99	**97.5**	**95**	**5**	**2.5**	**1**
1	0.000	0.001	0.004	3.841	5.024	6.635
2	0.020	0.051	0.103	5.992	7.378	9.210
3	0.115	0.216	0.352	7.815	9.348	11.345
4	0.297	0.484	0.711	9.488	11.143	13.277
5	0.554	0.831	1.145	11.070	12.833	15.086
6	0.872	1.237	1.635	12.592	14.449	16.812
7	1.239	1.690	2.167	14.067	16.013	18.475
8	1.646	2.180	2.733	15.507	17.535	20.090
9	2.088	2.700	3.325	16.919	19.023	21.666
10	2.558	3.247	3.940	18.307	20.483	23.209
11	3.053	3.816	4.575	19.675	21.920	24.725
12	3.571	4.404	5.226	21.026	23.337	26.217
13	4.107	5.009	5.892	22.362	24.736	27.688
14	4.660	5.629	6.571	23.685	26.119	29.141
15	5.229	6.262	7.261	24.996	27.488	30.578
16	5.812	6.908	7.962	26.296	28.845	32.000
17	6.408	7.564	8.672	27.587	30.191	33.409
18	7.015	8.231	9.390	28.869	31.526	34.805
19	7.633	8.907	10.117	30.144	32.852	36.191
20	8.260	9.591	10.851	31.410	34.170	37.566
25	11.524	13.120	14.611	37.652	40.646	44.314
30	14.953	16.791	18.493	43.773	46.979	50.892
40	22.164	24.433	26.509	55.758	59.342	63.691
50	29.707	32.357	34.764	67.505	71.420	76.154
80	53.540	57.153	60.391	101.879	106.629	112.329

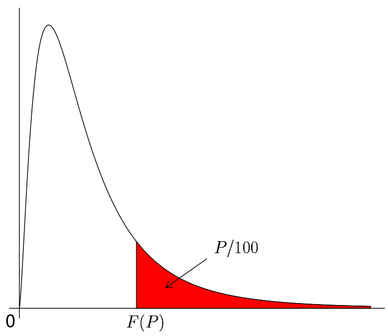
CONTINUED...

2.5 Percent Points of the F-Distribution

This table gives the percentage points $F_{\nu_1,\nu_2}(P)$ for $P = 0.025$ and degrees of freedom ν_1, ν_2, as indicated by the figure to the right.

The lower percentage points, that is the values $F'_{\nu_1,\nu_2}(P)$ such that the probability that $F \le F'_{\nu_1,\nu_2}(P)$ is equal to $P/100$, may be found using the formula

$$F'_{\nu_1,\nu_2}(P) = 1/F_{\nu_1,\nu_2}(P)$$



	ν_1								
ν_2	**1**	**2**	**3**	**4**	**5**	**6**	**12**	**24**	∞
2	38.506	39.000	39.165	39.248	39.298	39.331	39.415	39.456	39.498
3	17.443	16.044	15.439	15.101	14.885	14.735	14.337	14.124	13.902
4	12.218	10.649	9.979	9.605	9.364	9.197	8.751	8.511	8.257
5	10.007	8.434	7.764	7.388	7.146	6.978	6.525	6.278	6.015
6	8.813	7.260	6.599	6.227	5.988	5.820	5.366	5.117	4.849
7	8.073	6.542	5.890	5.523	5.285	5.119	4.666	4.415	4.142
8	7.571	6.059	5.416	5.053	4.817	4.652	4.200	3.947	3.670
9	7.209	5.715	5.078	4.718	4.484	4.320	3.868	3.614	3.333
10	6.937	5.456	4.826	4.468	4.236	4.072	3.621	3.365	3.080
11	6.724	5.256	4.630	4.275	4.044	3.881	3.430	3.173	2.883
12	6.554	5.096	4.474	4.121	3.891	3.728	3.277	3.019	2.725
13	6.414	4.965	4.347	3.996	3.767	3.604	3.153	2.893	2.595
14	6.298	4.857	4.242	3.892	3.663	3.501	3.050	2.789	2.487
15	6.200	4.765	4.153	3.804	3.576	3.415	2.963	2.701	2.395
16	6.115	4.687	4.077	3.729	3.502	3.341	2.889	2.625	2.316
17	6.042	4.619	4.011	3.665	3.438	3.277	2.825	2.560	2.247
18	5.978	4.560	3.954	3.608	3.382	3.221	2.769	2.503	2.187
19	5.922	4.508	3.903	3.559	3.333	3.172	2.720	2.452	2.133
20	5.871	4.461	3.859	3.515	3.289	3.128	2.676	2.408	2.085
25	5.686	4.291	3.694	3.353	3.129	2.969	2.515	2.242	1.906
30	5.568	4.182	3.589	3.250	3.026	2.867	2.412	2.136	1.787
40	5.424	4.051	3.463	3.126	2.904	2.744	2.288	2.007	1.637
50	5.340	3.975	3.390	3.054	2.833	2.674	2.216	1.931	1.545
100	5.179	3.828	3.250	2.917	2.696	2.537	2.077	1.784	1.347
∞	5.024	3.689	3.116	2.786	2.567	2.408	1.945	1.640	1.003

END