This question paper consists of 3 printed pages, each of which is identified by the reference MATH-2640

Only approved basic scientific calculators may be used.

Examination for the Module MATH-2640
(January 2007)

Introduction to Optimisation

Time allowed: **2 hours**

Attempt **four** questions. All questions carry equal marks.

Q1 **(a)** Define the gradient, ∇f, of a function $f(x, y, z)$. In the case

$$f(x, y, z) = x^2 + 2xy + z^3,$$

find ∇f. Find also the directional derivative in the direction

$$\mathbf{u} = \frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$$

at the point $x = y = z = 1$.

(b) $z(x, y)$ is defined implicitly by the relation

$$z^3 x - yz + xy^2 = 1.$$

Find the partial derivatives z_x, z_y and z_{xx} in terms of x, y and z. Find also all the possible points with $x = y = 1$ that satisfy this relation, and hence find all the possible values of z_x, z_y and z_{xx} with $x = y = 1$.

(c) Find the unit vector normal to the surface

$$f(x, y, z) = z^3 + x - 2y^3 = 0$$

at the point $x = y = z = 1$. Hence find the equation of the tangent plane to the surface $f = 0$ at $(x, y, z) = (1, 1, 1)$ in the form $ax + by + cz = d$.

(d) Given $u = u(x, y)$ and $v = v(x, y)$, define the Jacobian determinant that relates the differentials (du, dv) to the differentials (dx, dy). In the case

$$u = x^2 - y^2, \qquad v = x^2 + y^2,$$

find the conditions that the Jacobian determinant is zero, and explain why it is then not possible to find (dx, dy) in terms of (du, dv).

Q2 (a) Explain how the leading principal minors of an $n \times n$ matrix are defined.

(b) Find the symmetric matrix H for the quadratic form

$$Q(x, y, z) = 3x^2 + 3y^2 + 4z^2 + 4xy + 2xz + 2yz = \mathbf{x}^T H\mathbf{x},$$

and use the principal minor test to determine its sign properties.

(c) Given that $\lambda_1 = 1$ is an eigenvalue of the matrix H defined in part (b), find all its eigenvalues and normalized eigenvectors.

Hence write Q as a sum of three squares.

Q3 (a) Write down the Lagrangian, and hence find all the stationary points of the problem

$$f(x, y, z) = 2x + 4y + 3z^2,$$

subject to the constraint

$$h(x, y, z) = x^2 + y^2 + z^2 - 1 = 0.$$

(b) Find the 4×4 Bordered Hessian for the problem specified in part (a), and evaluate the two relevant leading principal minors. Hence classify all the the stationary points found in part (a).

Q4 (a) A rail company sells tickets to first class and economy class customers at prices p_1 and p_2 respectively. They can sell $Q_1 = 210 - p_1$ first class seats, and $Q_2 = 90 - p_2$ economy class seats. The cost to the company is

$$C = 6000 + Q_1^2 + Q_1 Q_2 + Q_2^2.$$

Find the prices p_1 and p_2 that give a stationary value for the profit and establish that it is a maximum. Can the company make a profit, or must it cut costs to avoid running at a loss?

(b) A company produces lampshades at a rate $x^{1/2}y^{1/2}$ which it sells for $£8$ per item. The inputs x and y are positive quantities. The cost of production is $C = x + 7y$ pounds. An equality constraint applies to x and y so that $x + y = 5$. Find the critical values of x and y that make the profit a stationary value, and the value of the profit at this stationary value. [You are not required to show that this stationary value is a maximum].

Q5 (a) A function $f(x, y)$ is to be maximised subject to the binding constraint $g(x, y) \leq b$. Explain by means of a sketch why ∇f and ∇g are parallel at a stationary point, and if $\nabla f = \lambda \nabla g$, state what sign λ must have.

(b) Use the Kuhn-Tucker method to maximise

$$f(x, y, z) = xyz + z,$$

subject to

$$x^2 + y^2 + z^2 \leq 25, \quad x \geq 0, \quad y \geq 0, \quad \text{and} \quad z \geq 0.$$

State also which constraints are binding at this maximum.

End.