This question paper consists of 3 printed pages, each of which is identified by the reference MATH-2640

Only approved basic scientific calculators may be used.

Examination for the Module MATH-2640
(January 2006)

Introduction to Optimisation

Time allowed: **2 hours**

Attempt **four** questions. All questions carry equal marks.

Q1 (a) Define the gradient, ∇f, of a function $f(x,y)$, and also the directional derivative of f in the direction of a unit vector $\mathbf{u}$. In the case

$$f(x,y) = x^2 + xy - 2y^2,$$

find both the gradient, ∇f, and the directional derivative in the direction

$$\mathbf{u} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$$

at the point $x = y = 1$. Also, write down a unit vector parallel to the level contour $f = 0$ at the point $x = y = 1$.

(b) If

$$f(x,y) = x^2y + 2xy^2,$$

with x and y constrained by the relation

$$y^2 = 3x^2 - 2xy,$$

find expressions for the partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$, and the total derivatives $\dfrac{df}{dx}$ and $\dfrac{df}{dy}$, in terms of x and y. Interpret $\dfrac{\partial f}{\partial x}$ and $\dfrac{df}{dx}$ geometrically.

(c) Find the unit vector normal to the surface

$$f(x,y,z) = z - 2x^2 - 3y^2 = 0$$

at the point $x = 1$, $y = 1$, $z = 5$. Hence find the equation of the tangent plane to the surface $z = 2x^2 + 3y^2$ at $(x,y,z) = (1,1,5)$ in the form $ax + by + cz = d$.

(d) The variables x, y and z are linked through the relations

$$f(x,y,z) = \sin x + \sin y + \sin z = 0,$$

$$g(x,y,z) = \cos x + \cos y + \cos z = 1.$$

Find $\dfrac{dy}{dx}$ in terms of x and y.

Q2 (a) Write down the first order conditions for $f(x, y, z)$ to have a stationary point.

(b) Define the three leading principal minors of a 3×3 matrix.

(c) Define the symmetric matrix H associated with the quadratic form

$$Q(x, y, z) = ax^2 + by^2 + cz^2 + 2exy + 2fxz + 2gyz,$$

and explain how the principal minors of H can be used to distinguish between positive definite, negative definite, positive semi-definite, negative semi-definite and indefinite quadratic forms.

(d) Use the leading principal minor test to determine the sign properties of the quadratic form

$$Q(x, y, z) = -2x^2 - y^2 - 5z^2 - 2xy + 4xz.$$

(e) Find all the stationary points of

$$f(x, y, z) = \frac{1}{4}x^4 + 2xy + \frac{3}{4}y^2 + yz + z^2.$$

(f) Find the Hessian matrix associated with the function $f(x, y, z)$, and hence classify all the stationary points.

Q3 (a) Write down the Lagrangian, and hence find the two stationary points of the problem

$$f(x, y, z) = \frac{1}{2}x^2 + xz + \frac{1}{2}y^2 + yz + \frac{5}{24}z^3,$$

subject to the constraint

$$x + y + z = 1.$$

(b) Explain what is meant by a Bordered Hessian, and write down (without proof) the conditions on the leading principal minors of the Bordered Hessian for a constrained stationary point to be a local maximum or a local minimum.

(c) Find the 4×4 Bordered Hessian for the problem specified in part (a), and evaluate the two relevant leading principal minors. Hence show that one of the two stationary points is a local minimum.

Continued ...

Q4 A firm supplies goods with a Cobb-Douglas production function

$$Q(x,y) = x^{\frac{1}{3}} y^{\frac{1}{3}}$$

and sells them at a price p each, where x and y are the input variables. The cost of production is $C = ax + by$. Write down the profit, Π, and show that it has a stationary point at

$$x = x^* = \frac{p^3}{27a^2b}, \qquad y = y^* = \frac{p^3}{27ab^2}.$$

Show, using Taylor's theorem for two variables, that if $x = x^* + h$ and $y = y^* + k$, then the profit

$$\Pi = \frac{p^3}{27ab} - \frac{9ab}{p^3}\mathbf{h}^T A\mathbf{h}, \quad \text{where } A = \begin{pmatrix} a^2 & -\frac{1}{2}ab \\ -\frac{1}{2}ab & b^2 \end{pmatrix} \text{ and } \mathbf{h}^T = (h,k),$$

provided cubic and higher order terms in h and k can be neglected.

In the case $a = b = 1$ find the eigenvalues and eigenvectors of the matrix A, and hence show that

$$\Pi = \frac{p^3}{27} - \frac{9}{p^3}\left[\frac{1}{4}(h+k)^2 + \frac{3}{4}(h-k)^2\right].$$

Deduce that the profit has a local maximum in the neighbourhood of (x^*, y^*).

Q5 (a) A function $f(x,y)$ is to be maximised subject to the binding constraint $g(x,y) \le b$. Explain by means of a sketch why ∇f and ∇g are parallel at a stationary point, and if $\nabla f = \lambda \nabla g$, state what sign λ must have.

(b) It is required to maximise

$$f = x^2 - 2xy + 2y^2,$$

subject to

$$x + y \le 1, \quad x \ge 0, \quad y \ge 0.$$

Write down the classical Lagrangian for this problem, $L(x, y, \lambda_1, \lambda_2, \lambda_3)$, where λ_1, λ_2 and λ_3 are the Lagrange multipliers, and state the five equality and six inequality conditions required to determine the stationary points. [*You are* **not** *required to find the stationary points by this method*].

(c) Now write down the *Kuhn-Tucker* Lagrangian $\bar{L}(x, y, \lambda)$ for this problem, and state the three equality conditions and six inequality conditions that determine the stationary points. Use these conditions to determine all the stationary points, and hence find the global maximum of f for this problem.

End.