MATH262001

This question paper consists of 5 printed pages, each of which is identified by the reference **MATH2620**.

Only approved basic scientific calculators may be used.

Examination for the Module MATH2620

(May 2007)

Fluid Dynamics

Time allowed: **2 hours**

Answer FOUR of the FIVE questions.

All questions carry equal marks.

1. (a) Explain what is meant by a particle path and a streamline. Under what circumstances are these the same?

(b) A two-dimensional flow is given by the velocity field

$$\boldsymbol{u} = \left(\frac{y}{b^2}, -\frac{(x - x_0)}{a^2} \right),$$

where x_0, a and b are positive constants.

Find the particle paths $(x(t), y(t))$ for this flow for the particle at $(2x_0, 0)$ at $t = 0$.

Show that this fluid flow is incompressible and calculate the corresponding streamfunction $\psi(x, t)$. Hence sketch the streamlines for this flow. Verify that for this flow the streamline through the point $(2x_0, 0)$ is the same as the particle path.

Write down the formula for the acceleration of a fluid particle, and hence calculate the fluid acceleration at a general point (x, y) at time t.

2. State the conditions under which the fluid velocity may be written as the gradient of a velocity potential, ϕ. Show that, when ϕ exists and the flow is incompressible, ϕ satisfies Laplace's equation.

An incompressible, two-dimensional, irrotational flow occupies the half-space $y < 0$. The potential satisfies Laplace's equation and the boundary conditions $\phi = 0$ at $x = 0$, 2π, and $\phi = \sin kx$ ($k > 0$) on $y = 0$, and $\phi \to 0$ as $y \to -\infty$. Using the method of separable solutions show that

$$\phi = \exp(ky) \sin(kx).$$

Calculate the velocity $\boldsymbol{u}(x, y)$ and the streamfunction $\psi(x, y)$. Sketch the streamlines for $k = 1$. Show that $(\nabla\phi) \cdot (\nabla\psi) = 0$.

3. Write down the equation of conservation of mass in Cartesian coordinates for an incompressible two-dimensional flow. State the relationship between the streamfunction $\psi(x, y)$ and the fluid velocity, and show that this satisfies the equation of conservation of mass. Show that ψ is constant along a streamline.

Define the vorticity of a velocity field, and show that for a two-dimensional flow

$$\nabla^2\psi = -\omega$$

where the vorticity $\boldsymbol{\omega} = (0, 0, \omega)$.

Two-dimensional flow is set-up between the two coaxial circular cylinders $r = a$ and $r = b$ ($b > a$). The vorticity distribution is given by

$$\omega = \frac{a^3}{4r} - r^2.$$

The azimuthal flow $u_\theta = 0$ on $r = a$ and $u_\theta = \Omega$ on $r = b$. On the assumption that the streamfunction is solely a function of radial distance, r, find the streamfunction and hence the velocity field in polar coordinates. Show that

$$\Omega = \frac{a^3 - b^3}{4}.$$

Calculate directly the circulation

$$\Gamma = \int \mathbf{u} \cdot d\mathbf{x}$$

around the circle $r = b$, and show that it is equal to

$$\int_S \omega dS,$$

where S is the area between $r = a$ and $r = b$.

4. Starting from Euler's equation in the form

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho}\nabla p + \mathbf{g}$$

show that for a steady flow

$$\frac{p}{\rho} + gz + \frac{1}{2}u^2$$

is constant along a streamline, where z is the upward vertical coordinate. (You may quote any of the vector identities at the end of the examination paper).

Show, in addition, that for a steady irrotational flow ($\boldsymbol{\omega} = 0$) that

$$\frac{p}{\rho} + gz + \frac{1}{2}u^2$$

is constant everywhere in the flow.

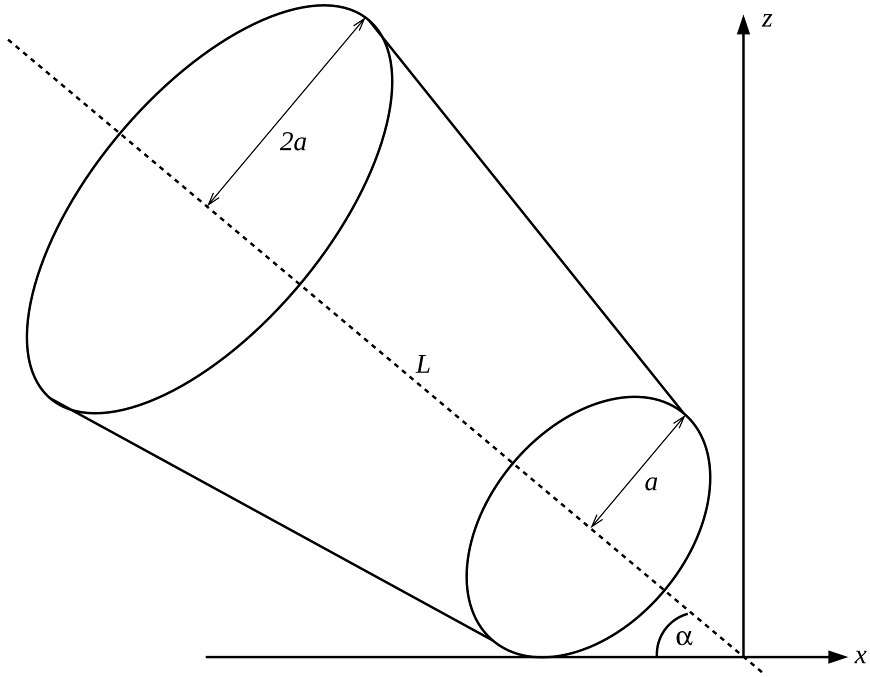

Figure 1: Section of the pipe for Question 4

A long straight pipe of length L has a slowly varying circular cross-section. It is inclined so that its axis is at an angle α to the horizontal, with its smaller cross-section downwards (as shown in the figure). The radius of the pipe at its upper end is $2a$ whilst that at the lower end is a. Water (with density ρ) is pumped at a steady rate through the pipe such that the pressure at the top of the pipe is $2p_a$ and that at the bottom is atmospheric pressure p_a. Show that the water emerges from the pipe with speed U given by

$$U^2 = \frac{32}{15}\left(gL\sin\alpha + \frac{p_a}{\rho}\right),$$

and find the velocity half-way along the pipe.

5. Water flows in a long horizontal channel. The breadth, b, of the channel is equal to B except for an intermediate section where the channel narrows gradually to a minimum width B_m before gradually widening again to B. Upstream the fluid velocity is is U and the water is of depth H.

Write down relations obtained from the conservation of mass and Bernoulli's equation.

Define the Froude number, and show that if the upstream Froude number is $\sqrt{3/5}$ then the breadth of the channel, b, and the height of fluid, h, are related by

$$\frac{B^2}{b^2} = \frac{h^2}{H^2}\left(\frac{13}{3} - \frac{10h}{3H}\right)$$

Show that the height of the flow downstream of the constriction must be either $\frac{h}{H} = 1$ or $\frac{h}{H} = \frac{3+\sqrt{129}}{20}$. In the latter case calculate the depth and velocity of the water in the channel at the narrowest point.

Formulae Sheet

Useful Vector Identities

$$
\begin{aligned}
\nabla \times \nabla p &= 0, \\
\nabla\cdot (\nabla \times \boldsymbol{u}) &= 0, \\
\nabla\cdot (p\boldsymbol{u}) &= p\nabla\cdot\boldsymbol{u} + \boldsymbol{u}\cdot\nabla p, \\
\nabla \times (p\boldsymbol{u}) &= p\nabla \times \boldsymbol{u} + \nabla p \times \boldsymbol{u}, \\
\nabla \times (\mathbf{B} \times \mathbf{A}) &= \mathbf{A}\cdot\nabla\mathbf{B} - \mathbf{B}\cdot\nabla\mathbf{A} + \mathbf{A}\nabla\cdot\mathbf{B} - \mathbf{B}\nabla\cdot\mathbf{A}, \\
\nabla\cdot (\mathbf{A} \times \mathbf{B}) &= \mathbf{B}\cdot\nabla \times \mathbf{A} - \mathbf{A}\cdot\nabla \times \mathbf{B}, \\
\nabla (\mathbf{A}\cdot\mathbf{B}) &= \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + \mathbf{A}\cdot\nabla\mathbf{B} + \mathbf{B}\cdot\nabla\mathbf{A}, \\
\nabla^2\boldsymbol{u} &= \nabla (\nabla\cdot\boldsymbol{u}) - \nabla \times (\nabla \times \boldsymbol{u}), \\
(\nabla \times \boldsymbol{u}) \times \boldsymbol{u} &= \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nabla \left(\tfrac{1}{2}\boldsymbol{u}^2\right).
\end{aligned}
$$

Cartesian coordinates

Scalar p, vector $\boldsymbol{u} = u\mathbf{e}_x + v\mathbf{e}_y + w\mathbf{e}_z$

$$
\begin{aligned}
\mathrm{grad} p = \nabla p &= \frac{\partial p}{\partial x}\mathbf{e}_x + \frac{\partial p}{\partial y}\mathbf{e}_y + \frac{\partial p}{\partial z}\mathbf{e}_z, \\
\mathrm{div}\boldsymbol{u} = \nabla\cdot\boldsymbol{u} &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}, \\
\mathrm{curl}\boldsymbol{u} = \nabla \times \boldsymbol{u} &= \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\mathbf{e}_x + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\mathbf{e}_y + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{e}_z, \\
\boldsymbol{u}\cdot\nabla p &= u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}, \\
\nabla^2 p &= \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}, \\
\boldsymbol{u}\cdot\nabla\boldsymbol{u} &= \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)\mathbf{e}_x + \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)\mathbf{e}_y \\
&\quad + \left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)\mathbf{e}_z
\end{aligned}
$$

CONTINUED...

Cylindrical Polar Coordinates

$\boldsymbol{u} = u\mathbf{e}_r + v\mathbf{e}_\theta + w\mathbf{e}_z$

$$\nabla p = \frac{\partial p}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\mathbf{e}_\theta + \frac{\partial p}{\partial z}\mathbf{e}_z,$$

$$\nabla\cdot\boldsymbol{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z},$$

$$\nabla\times\boldsymbol{u} = \left(\frac{1}{r}\frac{\partial w}{\partial \theta} - \frac{\partial v}{\partial z}\right)\mathbf{e}_r + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right)\mathbf{e}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}(rv) - \frac{\partial u}{\partial \theta}\right)\mathbf{e}_z,$$

$$\boldsymbol{u}\cdot\nabla p = u\frac{\partial p}{\partial r} + \frac{v}{r}\frac{\partial p}{\partial \theta} + w\frac{\partial p}{\partial z},$$

$$\nabla^2 p = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} + \frac{\partial^2 p}{\partial z^2}.$$

Spherical Polar Coordinates

$\boldsymbol{u} = u\mathbf{e}_r + v\mathbf{e}_\theta + w\mathbf{e}_\phi$

$$\nabla p = \frac{\partial p}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}\mathbf{e}_\phi,$$

$$\nabla\cdot\boldsymbol{u} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(v\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial w}{\partial \phi},$$

$$\nabla\times\boldsymbol{u} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \theta}(w\sin\theta) - \frac{\partial v}{\partial \phi}\right)\mathbf{e}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial u}{\partial \phi} - \frac{\partial}{\partial r}(rw)\right)\mathbf{e}_\theta$$
$$+ \frac{1}{r}\left(\frac{\partial}{\partial r}(rv) - \frac{\partial u}{\partial \theta}\right)\mathbf{e}_\phi,$$

$$\boldsymbol{u}\cdot\nabla p = u\frac{\partial p}{\partial r} + \frac{v}{r}\frac{\partial p}{\partial \theta} + \frac{w}{r\sin\theta}\frac{\partial p}{\partial \phi},$$

$$\nabla^2 p = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial p}{\partial \theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2 p}{\partial \phi^2}.$$

Divergence Theorem and Stokes Theorem

Let V be a region bounded by a simple closed surface S with unit **outward** normal **n**

$$\int_S \boldsymbol{u}\cdot\mathbf{n}\,\mathrm{d}S = \int_V \nabla\cdot\boldsymbol{u}\,\mathrm{d}V, \qquad \int_S p\mathbf{n}\,\mathrm{d}S = \int_V \nabla p\, dV, \qquad \int_S \boldsymbol{u}\times\mathbf{n}\,\mathrm{d}S = -\int_V \nabla\times\boldsymbol{u}\, dV.$$

Let C be a simple closed curve spanned by a surface S with unit normal **n**

$$\int_C \boldsymbol{u}\cdot\mathrm{d}\mathbf{x} = \int_S (\nabla\times\boldsymbol{u})\cdot\mathbf{n}\,\mathrm{d}S, \qquad \int_C p\,\mathrm{d}\mathbf{x} = -\int_S (\nabla p)\times\mathbf{n}\,\mathrm{d}S.$$

END