MATH262001

This question paper consists of 6 printed pages, each of which is identified by the reference **MATH2620**.

Only approved basic scientific calculators may be used.

Examination for the Module MATH2620

(May 2006)

Fluid Dynamics

Time allowed: **2 hours**

Answer FOUR of the FIVE questions.

All questions carry equal marks.

1. Explain what is meant by a particle path and a streamline. Under what circumstances are these the same?

A two-dimensional flow is given by the velocity field

$$\boldsymbol{u} = (x\,(1+t)\,,y)$$

Find the particle paths $(x(t), y(t))$ for this flow for the particle at $(1,1)$ at $t = 0$. Hence show that $y(1) = e^{-\frac{1}{2}}x(1)$. Find and sketch the streamline that passes through the point $(1,1)$ at $t = 0$.

Calculate $\boldsymbol{\nabla}\cdot\boldsymbol{u}$ for this flow and hence show that the flow is *not* incompressible.

Write down the formula for the acceleration of a fluid particle, and hence calculate the fluid acceleration at a general point (x, y) at time t.

2. State the conditions under which the fluid velocity may be written as the gradient of a velocity potential, ϕ. Show that, when ϕ exists and the flow is incompressible, ϕ satisfies Laplace's equation.

Separable solutions of Laplace's equation exist in plane-polar co-ordinates (r, θ) of the form $\phi(r, \theta) = f(r)\cos(\theta)$. Find the general solutions for $f(r)$.

Hence show that the potential given by

$$\phi(r, \theta) = \left(Ur + \frac{Ua^2}{r}\right)\cos\theta$$

represents the unique potential for flow around a cylinder of radius a with uniform flow U in the x-direction at $x = \pm\infty$. (Hint: show that this potential does satisfy Laplace's equation and the required boundary conditions on the surface of the cylinder and at infinity.)

Show that

$$u_\theta = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta.$$

By integrating this around a circle of radius $b > a$ show that the circulation around the cylinder

$$\Gamma = \oint \mathbf{u} \cdot d\mathbf{x}$$

is zero.

3. Define the vorticity $\boldsymbol{\omega}$ of a fluid with velocity $\boldsymbol{u}$. Show that for two-dimensional incompressible flows where $\boldsymbol{u} = (u, v, 0)$ is independent of z

$$\boldsymbol{\omega} = \left(0, 0, -\nabla^2\psi\right) \qquad (1)$$

where ψ is the streamfunction that you should relate to u and v.

Consider the flow $\boldsymbol{u} = u_r\hat{\boldsymbol{r}} + u_\theta\hat{\boldsymbol{\theta}}$ in cylindrical polars (r, θ, z) where

$$u_r = \frac{Ua^2}{r^2}\cos\theta, \qquad u_\theta = \frac{Ua^2}{r^2}\sin\theta,$$

Calculate $\boldsymbol{\omega}$ and the streamfunction ψ and verify the relationship derived in equation (1).

Starting from Euler's equation in the form

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\frac{1}{\rho}\nabla p,$$

derive the vorticity equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{\omega} = \boldsymbol{\omega}\cdot\nabla\boldsymbol{u}.$$

Hence show that the vorticity of a fluid element does not change as it moves in a planar flow.

4. Starting from Euler's equation in the form

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\frac{1}{\rho}\nabla p + \mathbf{g}$$

show that for a steady flow

$$\frac{p}{\rho} + gz + \frac{1}{2}u^2$$

is constant along a streamline, where z is the upward vertical coordinate. (You may quote any of the vector identities at the end of the examination paper).

Show, in addition, that for a steady irrotational flow ($\boldsymbol{\omega} = 0$) that

$$\frac{p}{\rho} + gz + \frac{1}{2}u^2$$

 QUESTION 4 CONTINUED...

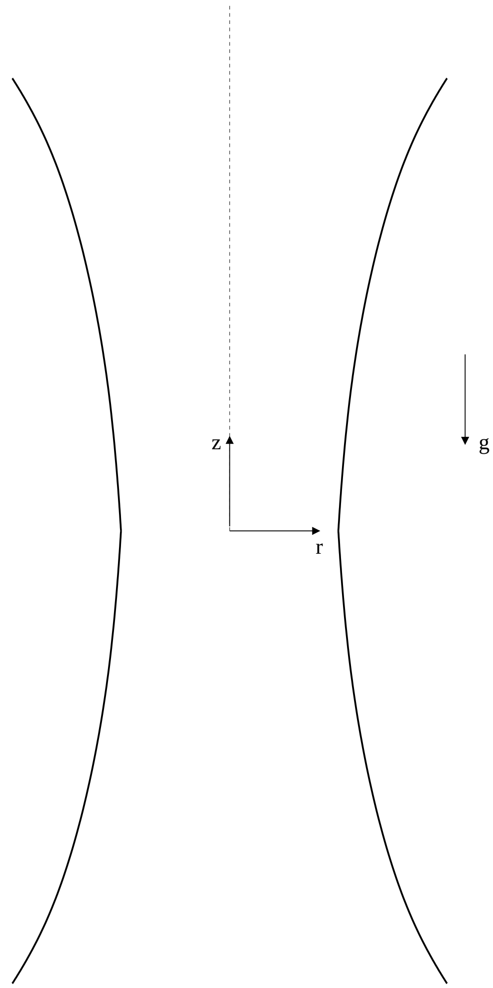

Figure 1: Section of the pipe for Question 4

is constant everywhere in the flow.

An axisymmetric vertical pipe of variable cross-section (as shown in Figure 1) is filled with water. The cross-section is given by $r = a(\cosh \alpha z)^{\frac{1}{4}}$, where a and α are positive constants, (r, θ, z) are cylindrical polar co-ordinates with the z-axis vertical, and gravity acting downwards so $\mathbf{g} = (0, 0, -g)$. The water is incompressible and is in steady irrotational flow along the pipe, with a volume Q of fluid passing every cross-section per unit time. Assuming that the vertical component of velocity u_z depends on z only and that horizontal velocities are small and can be neglected, calculate u_z as a function of z.

Hence or otherwise show that the water pressure will not be a monotonic function of z if

$$\alpha Q^2 > 4\pi^2 a^4 g.$$

5. Water flows in the x-direction in a long channel of rectangular cross-section, whose base lies on the plane $z = 0$ except for a small smooth hump whose maximum height is d_m. The equation of the hump is $z = d(x)$. The flow is steady and smooth throughout the channel. Far upstream of the hump the flow has velocity U and the surface of the water lies at $z = H$.

QUESTION 5 CONTINUED...

Calculate the level of the water surface above the bump (denoted $\delta(x)$, where $\delta(x)$ is the height of the water above the value it would assume if there were no bump) when $H \gg d_m$ demonstrating that it is given by the equation

$$\delta = \frac{U^2}{2g}\left(1 - \left(\frac{H}{H+\delta-d}\right)^2\right).$$

Show that for small d and δ the surface may rise or fall as it flows over the bump depending upon the value of the upstream Froude number, $F = \dfrac{U}{\sqrt{gH}}$.

Hint: you may use the expansion

$$\left(\frac{H}{H+\delta-d}\right)^2 \approx 1 - \frac{2(\delta-d)}{H}.$$

Far downstream of the hump the water level is at height $7H/16$. Show that the upstream Froude number, F, must be equal to $\left(\frac{49}{184}\right)^{1/2}$, and find the value far downstream. At what point is the Froude number equal to unity?

Formulae Sheet

Useful Vector Identities

$$
\begin{aligned}
\nabla \times \nabla p &= 0, \\
\nabla\cdot (\nabla \times \boldsymbol{u}) &= 0, \\
\nabla\cdot (p\boldsymbol{u}) &= p\nabla\cdot\boldsymbol{u} + \boldsymbol{u}\cdot\nabla p, \\
\nabla \times (p\boldsymbol{u}) &= p\nabla \times \boldsymbol{u} + \nabla p \times \boldsymbol{u}, \\
\nabla \times (\mathbf{B} \times \mathbf{A}) &= \mathbf{A}\cdot\nabla\mathbf{B} - \mathbf{B}\cdot\nabla\mathbf{A} + \mathbf{A}\nabla\cdot\mathbf{B} - \mathbf{B}\nabla\cdot\mathbf{A}, \\
\nabla\cdot (\mathbf{A} \times \mathbf{B}) &= \mathbf{B}\cdot\nabla \times \mathbf{A} - \mathbf{A}\cdot\nabla \times \mathbf{B}, \\
\nabla (\mathbf{A}\cdot\mathbf{B}) &= \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + \mathbf{A}\cdot\nabla\mathbf{B} + \mathbf{B}\cdot\nabla\mathbf{A}, \\
\nabla^2\boldsymbol{u} &= \nabla (\nabla\cdot\boldsymbol{u}) - \nabla \times (\nabla \times \boldsymbol{u}), \\
(\nabla \times \boldsymbol{u}) \times \boldsymbol{u} &= \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nabla \left(\tfrac{1}{2}\boldsymbol{u}^2\right).
\end{aligned}
$$

Cartesian coordinates

Scalar p, vector $\boldsymbol{u} = u\mathbf{e}_x + v\mathbf{e}_y + w\mathbf{e}_z$

$$
\begin{aligned}
\mathrm{grad} p = \nabla p &= \frac{\partial p}{\partial x}\mathbf{e}_x + \frac{\partial p}{\partial y}\mathbf{e}_y + \frac{\partial p}{\partial z}\mathbf{e}_z, \\
\mathrm{div}\boldsymbol{u} = \nabla\cdot\boldsymbol{u} &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}, \\
\mathrm{curl}\boldsymbol{u} = \nabla \times \boldsymbol{u} &= \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\mathbf{e}_x + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\mathbf{e}_y + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{e}_z, \\
\boldsymbol{u}\cdot\nabla p &= u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}, \\
\nabla^2 p &= \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}, \\
\boldsymbol{u}\cdot\nabla\boldsymbol{u} &= \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)\mathbf{e}_x + \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)\mathbf{e}_y \\
&\quad + \left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)\mathbf{e}_z
\end{aligned}
$$

CONTINUED...

Cylindrical Polar Coordinates

$\boldsymbol{u} = u\mathbf{e}_r + v\mathbf{e}_\theta + w\mathbf{e}_z$

$$\begin{aligned}
\nabla p &= \frac{\partial p}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\mathbf{e}_\theta + \frac{\partial p}{\partial z}\mathbf{e}_z, \\
\nabla\cdot\boldsymbol{u} &= \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z}, \\
\nabla\times\boldsymbol{u} &= \left(\frac{1}{r}\frac{\partial w}{\partial \theta} - \frac{\partial v}{\partial z}\right)\mathbf{e}_r + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right)\mathbf{e}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}(rv) - \frac{\partial u}{\partial \theta}\right)\mathbf{e}_z, \\
\boldsymbol{u}\cdot\nabla p &= u\frac{\partial p}{\partial r} + \frac{v}{r}\frac{\partial p}{\partial \theta} + w\frac{\partial p}{\partial z}, \\
\nabla^2 p &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} + \frac{\partial^2 p}{\partial z^2}.
\end{aligned}$$

Spherical Polar Coordinates

$\boldsymbol{u} = u\mathbf{e}_r + v\mathbf{e}_\theta + w\mathbf{e}_\phi$

$$\begin{aligned}
\nabla p &= \frac{\partial p}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}\mathbf{e}_\phi, \\
\nabla\cdot\boldsymbol{u} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(v\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial w}{\partial \phi}, \\
\nabla\times\boldsymbol{u} &= \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \theta}(w\sin\theta) - \frac{\partial v}{\partial \phi}\right)\mathbf{e}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial u}{\partial \phi} - \frac{\partial}{\partial r}(rw)\right)\mathbf{e}_\theta \\
&\quad + \frac{1}{r}\left(\frac{\partial}{\partial r}(rv) - \frac{\partial u}{\partial \theta}\right)\mathbf{e}_\phi, \\
\boldsymbol{u}\cdot\nabla p &= u\frac{\partial p}{\partial r} + \frac{v}{r}\frac{\partial p}{\partial \theta} + \frac{w}{r\sin\theta}\frac{\partial p}{\partial \phi}, \\
\nabla^2 p &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial p}{\partial \theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2 p}{\partial \phi^2}.
\end{aligned}$$

Divergence Theorem and Stokes Theorem

Let V be a region bounded by a simple closed surface S with unit **outward** normal **n**

$$\int_S \boldsymbol{u}\cdot\mathbf{n}\,\mathrm{d}S = \int_V \nabla\cdot\boldsymbol{u}\,\mathrm{d}V, \qquad \int_S p\mathbf{n}\,\mathrm{d}S = \int_V \nabla p\,dV, \qquad \int_S \boldsymbol{u}\times\mathbf{n}\,\mathrm{d}S = -\int_V \nabla\times\boldsymbol{u}\,dV.$$

Let C be a simple closed curve spanned by a surface S with unit normal **n**

$$\int_C \boldsymbol{u}\cdot\mathrm{d}\mathbf{x} = \int_S (\nabla\times\boldsymbol{u})\cdot\mathbf{n}\,\mathrm{d}S, \qquad \int_C p\,\mathrm{d}\mathbf{x} = -\int_S (\nabla p)\times\mathbf{n}\,\mathrm{d}S.$$

END