MATH262001

This question paper consists of 5 printed pages, each of which is identified by the reference **MATH2620**.

Only approved basic scientific calculators may be used.

Examination for the Module MATH2620

(May 2005)

Fluid Dynamics

Time allowed: **2 hours**

Answer FOUR of the FIVE questions.

All questions carry equal marks.

1. Explain what is meant by a particle path and streamline. Under what circumstances are these the same?

A two-dimensional flow is given by the velocity field

$$\mathbf{u} = (\sin t, x)$$

Find the particle paths for this flow for the particle initially at (x_0, y_0). Hence show that for $x_0 = -1$ the particle paths have the form of circles centred on $(0, y_0)$. Find and sketch the streamline that passes through the point $(1, 0)$ at $t = \pi/2$.

Show that $\boldsymbol{\nabla}\cdot\mathbf{u} = 0$ for this flow and find the corresponding streamfunction of this flow at time t.

Write down the formula for the acceleration of a fluid particle, and hence calculate the fluid acceleration at a general point (x, y) at time t.

2. State the conditions under which the fluid velocity may be written as the gradient of a velocity potential, ϕ? Show that, when ϕ exists and the flow is incompressible, ϕ satisfies Laplace's equation.

By seeking a solution of Laplace's equation of the form $\phi(r)$ (where r is radial distance from the origin in spherical polars) find the potential for a point source for which the radial velocity is $K\,\mathbf{e}_r$ on the sphere $r = R_0$, where K is a constant.

A point source is located at a distance d from a plane wall ($x = 0$). By constructing an appropriate image system, find the velocity potential for this flow. Calculate $u = \partial\phi/\partial x$ and show that the boundary conditions are satisfied at the wall. Further calculate $v = \partial\phi/\partial y$ and $w = \partial\phi/\partial z$ at $x = 0$

Hence calculate the speed of the flow at the point $\left(0, \dfrac{\sqrt{3}d}{2}, \dfrac{\sqrt{3}d}{2}\right)$, where d is a constant.

3. Write down the equation of conservation of mass in Cartesian coordinates for an incompressible two-dimensional flow. State the relationship between the streamfunction $\psi(x, y)$ and the fluid velocity, and show that this satisfies the equation of conservation of mass. Show that ψ is constant along a streamline.

Define the vorticity of a velocity field, and show that for a two-dimensional flow

$$\nabla^2\psi = -\omega$$

where ω is the magnitude of the vorticity.

Two-dimensional flow is set-up inside the circular cylinder $r = a$ with a vorticity distribution

$$\omega = r^2 - \frac{3ar}{4}$$

On the assumption that the streamfunction is solely a function of radial distance, r, find the streamfunction and hence the velocity field in polar coordinates. Show that the velocity is zero on the surface of the cylinder $r = a$.

Calculate directly the circulation

$$\Gamma = \int \mathbf{u}\cdot d\mathbf{x}$$

around the circle $r = a$ and show that it is equal to

$$\int_S \omega dS,$$

where S is the interior of the circle of radius a.

4. Starting from Euler's equation in the form

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p + \mathbf{g}$$

show that for a steady flow

$$\frac{p}{\rho} + gz + \frac{1}{2}u^2$$

is constant along a streamline, where z is the upward vertical coordinate. (You may quote any of the vector identities at the end of the examination paper).

Show, in addition, that for a steady irrotational flow ($\boldsymbol{\omega} = 0$) that

$$\frac{p}{\rho} + gz + \frac{1}{2}u^2$$

is constant everywhere in the flow.

A pump is designed as shown in Figure 1. Water flows along the horizontal pipe, through a contraction and out into the atmosphere soon after. For fast enough flows the water may be sucked up through the thin vertical pipe and out of the reservoir. The volume flow rate is Q and the cross-sectional areas are A_1 in the pipe and A_2 in the contraction.

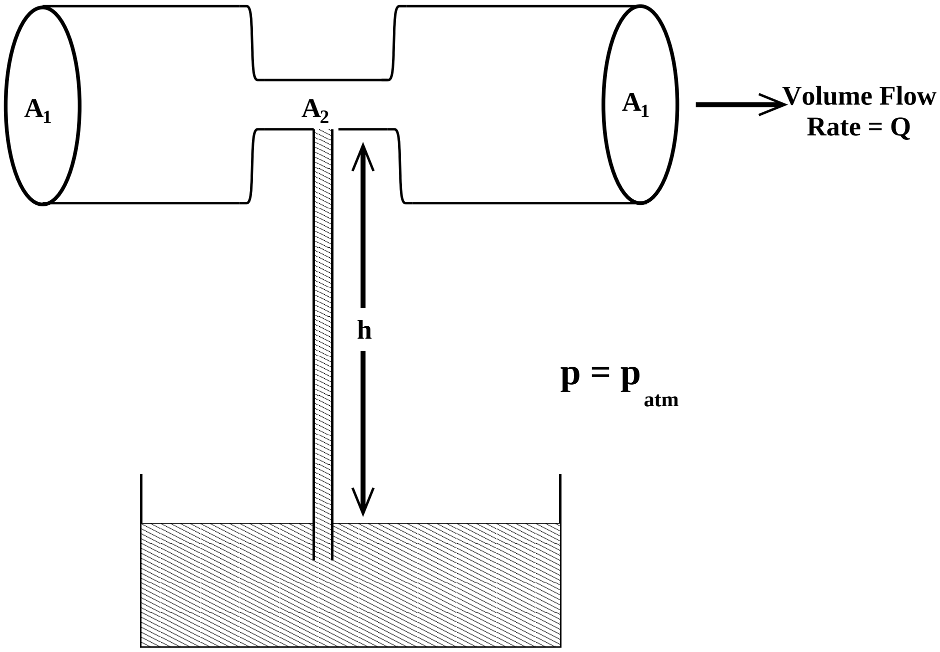

Figure 1: Pump for Question 4

Derive the pressure in the contraction in terms of the volume flow-rate, the atmospheric pressure and the cross-sectional areas.

Derive a condition on the pressure at the top of the thin pipe for fluid to be sucked out of the reservoir.

Hence show that fluid is sucked upwards if

$$h < \frac{Q^2}{2gA_1^2A_2^2}\left(A_1^2 - A_2^2\right).$$

5. Water flows in a long horizontal channel. The breadth, b, of the channel is equal to B except for an intermediate section where the channel narrows gradually to a minimum width B_m before gradually widening again to B. Upstream the fluid velocity is is U and the water is of depth H.

Write down relations obtained from the conservation of mass and Bernoulli's equation.

Define the Froude number, and show that if the upstream Froude number is $\sqrt{2/5}$ then the breadth of the channel, b and the height of fluid, h are related by

$$\frac{B^2}{b^2} = \frac{h^2}{H^2}\left(6 - \frac{5h}{H}\right)$$

Show that the height of the flow downstream of the constriction must be either $h = 1$ or $h = \frac{1+\sqrt{21}}{10}$. In the latter case calculate the breadth and depth of the water of the channel at the narrowest point.

Formulae Sheet

Useful Vector Identities

$$
\begin{aligned}
\nabla \times \nabla p &= 0, \\
\nabla\cdot(\nabla \times \mathbf{u}) &= 0, \\
\nabla\cdot(p\mathbf{u}) &= p\nabla\cdot\mathbf{u} + \mathbf{u}\cdot\nabla p, \\
\nabla \times (p\mathbf{u}) &= p\nabla \times \mathbf{u} + \nabla p \times \mathbf{u}, \\
\nabla \times (\mathbf{A} \times \mathbf{B}) &= \mathbf{A}\cdot\nabla\mathbf{B} - \mathbf{B}\cdot\nabla\mathbf{A} + \mathbf{A}\nabla\cdot\mathbf{B} - \mathbf{B}\nabla\cdot\mathbf{A}), \\
\nabla\cdot(\mathbf{A} \times \mathbf{B}) &= \mathbf{B}\cdot\nabla \times \mathbf{A} - \mathbf{A}\cdot\nabla \times \mathbf{B}, \\
\nabla(\mathbf{A}\cdot\mathbf{B}) &= \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + \mathbf{A}\cdot\nabla\mathbf{B} + \mathbf{B}\cdot\nabla\mathbf{A}, \\
\nabla^2\mathbf{u} &= \nabla(\nabla\cdot\mathbf{u}) - \nabla \times (\nabla \times \mathbf{u}), \\
(\nabla \times \mathbf{u}) \times \mathbf{u} &= \mathbf{u}\cdot\nabla\mathbf{u} - \nabla\left(\tfrac{1}{2}\mathbf{u}^2\right).
\end{aligned}
$$

Cartesian coordinates

Scalar p, vector $\mathbf{u} = u\mathbf{e}_x + v\mathbf{e}_y + w\mathbf{e}_z$

$$
\begin{aligned}
\mathrm{grad}p = \nabla p &= \frac{\partial p}{\partial x}\mathbf{e}_x + \frac{\partial p}{\partial y}\mathbf{e}_y + \frac{\partial p}{\partial z}\mathbf{e}_z, \\
\mathrm{div}\mathbf{u} = \nabla\cdot\mathbf{u} &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}, \\
\mathrm{curl}\mathbf{u} = \nabla \times \mathbf{u} &= \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\mathbf{e}_x + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\mathbf{e}_y + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{e}_z, \\
\mathbf{u}\cdot\nabla p &= u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}, \\
\nabla^2 p &= \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}, \\
\mathbf{u}\cdot\nabla\mathbf{u} &= \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)\mathbf{e}_x + \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)\mathbf{e}_y \\
&\quad + \left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)\mathbf{e}_z
\end{aligned}
$$

CONTINUED...

Cylindrical Polar Coordinates

$\mathbf{u} = u\mathbf{e}_r + v\mathbf{e}_\theta + w\mathbf{e}_z$

$$\begin{aligned}
\nabla p &= \frac{\partial p}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\mathbf{e}_\theta + \frac{\partial p}{\partial z}\mathbf{e}_z, \\
\nabla\cdot\mathbf{u} &= \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z}, \\
\nabla\times\mathbf{u} &= \left(\frac{1}{r}\frac{\partial w}{\partial \theta} - \frac{\partial v}{\partial z}\right)\mathbf{e}_r + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right)\mathbf{e}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}(rv) - \frac{\partial u}{\partial \theta}\right)\mathbf{e}_z, \\
\mathbf{u}\cdot\nabla p &= u\frac{\partial p}{\partial r} + \frac{v}{r}\frac{\partial p}{\partial \theta} + w\frac{\partial p}{\partial z}, \\
\nabla^2 p &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} + \frac{\partial^2 p}{\partial z^2}.
\end{aligned}$$

Spherical Polar Coordinates

$\mathbf{u} = u\mathbf{e}_r + v\mathbf{e}_\theta + w\mathbf{e}_\phi$

$$\begin{aligned}
\nabla p &= \frac{\partial p}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}\mathbf{e}_\phi, \\
\nabla\cdot\mathbf{u} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(v\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial w}{\partial \phi}, \\
\nabla\times\mathbf{u} &= \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \theta}(w\sin\theta) - \frac{\partial v}{\partial \phi}\right)\mathbf{e}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial u}{\partial \phi} - \frac{\partial}{\partial r}(rw)\right)\mathbf{e}_\theta \\
&\quad + \frac{1}{r}\left(\frac{\partial}{\partial r}(rv) - \frac{\partial u}{\partial \theta}\right)\mathbf{e}_\phi, \\
\mathbf{u}\cdot\nabla p &= u\frac{\partial p}{\partial r} + \frac{v}{r}\frac{\partial p}{\partial \theta} + \frac{w}{r\sin\theta}\frac{\partial p}{\partial \phi}, \\
\nabla^2 p &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial p}{\partial \theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2 p}{\partial \phi^2}.
\end{aligned}$$

Divergence Theorem and Stokes Theorem

Let V be a region bounded by a simple closed surface S with unit **outward** normal **n**

$$\int_S \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S = \int_V \nabla\cdot\mathbf{u}\,\mathrm{d}V, \qquad \int_S p\mathbf{n}\,\mathrm{d}S = \int_V \nabla p\,dV, \qquad \int_S \mathbf{u}\times\mathbf{n}\,\mathrm{d}S = -\int_V \nabla\times\mathbf{u}\,dV.$$

Let C be a simple closed curve spanned by a surface S with unit normal **n**

$$\int_C \mathbf{u}\cdot\mathrm{d}\mathbf{x} = \int_S (\nabla\times\mathbf{u})\cdot\mathbf{n}\,\mathrm{d}S, \qquad \int_C p\,\mathrm{d}\mathbf{x} = -\int_S (\nabla p)\times\mathbf{n}\,\mathrm{d}S.$$

END