**MATH260001**

This question paper consists of 3 printed pages, each of which is identified by the reference **MATH260001**.

Only approved basic scientific calculators may be used.



Examination for the Module MATH2600

(January 2007)

**Numerical Analysis**

Time allowed: **2 hours**

Answer FOUR of the FIVE questions.

All questions carry equal marks.

**1.** (a) Show that the function $f(x) = \exp(x) - \frac{3}{2}$ has only one zero. Write down the general step of Newton's method for solving $f(x) = 0$. Starting with the an initial guess of $p_0 = 0$, calculate the next three approximations to the solution of $f(x) = \exp(x) - \frac{3}{2} = 0$ using Newton's method. For each iteration calculate the error from the exact solution.

(b) Write down the general step for solving $f(x) = \exp(x) - \frac{3}{2} = 0$ using the Secant method and taking $p_0 = 0$ and $p_1 = 0.5$ calculate two further approximations. For each iteration calculate the error from the exact solution.

(c) Write down the map $p_{n+1} = g(p_n)$ corresponding to the Newton's method solution of $f(x) = \exp(x) - \frac{3}{2} = 0$. Show that $g'(x) = 0$ at the fixed point. Sketch a plot of $g(x)$ for $x > 0$.

State the fixed point theorem giving the conditions that guarantee the iteration scheme $x_{n+1} = g(x_n)$ has a unique fixed point in the interval $[a, b]$. Show that $g(x)$ satisfies the conditions of this theorem in the interval [0,0.5].

**2.** (a) Write down the Lagrange form of the interpolating polynomial $P(x)$ that satisfies

$$P(x) = f(x) \qquad \text{for } x = x_1, x_2 \ldots x_n.$$

Find the form of the polynomial that interpolates $f(x) = \frac{1}{x}$ through the points $x = 0.5,\ 1$ and $3$. (You do not need to simplify it). Use this polynomial to estimate the value of $f(2)$ and find the error from the actual value.

Using the error formula for Lagrange interpolation,

$$f(x) = P(x) + \frac{1}{n!}\prod_{i=1}^{n}(x - x_i)\frac{d^n f}{dx^n}(\xi)\,, \qquad \text{for some } \xi \in (x_1, x_n),$$

find upper and lower bounds for the error in your estimate above. How does this compare with the actual error?

Find the form of the error term when there are $n$ interpolation points in the interval [0.5,3]. Use this formula to calculate error bounds on the estimate of $f(2)$ obtained from Lagrange interpolation of $f(x) = 1/x$ through the points $0.5$, $0.75$, $1$, $1.25$, $1.5$ and $3$. What happens to error bounds as you increase the number of interpolation points for this function?

**3.** (a) Simpson's rule for integrating a function over the interval $[-h, h]$ can be written in the form,

$$\int_{-h}^{h} f(x)dx = w_1 f(-h) + w_2 f(0) + w_3 f(h)$$

where $w_1$, $w_2$ and $w_3$ are constants. Use the method of undetermined coefficients to find the constants $w_1$, $w_2$ and $w_3$. Calculate the Simpson's rule approximation to the integral

$$I = \int_{0}^{\pi/2} \cos(x)dx,$$

and calculate the error from the actual answer.

(b) Show that Simpson's rule is exact for all cubic polynomials and hence using a suitable quartic polynomial obtain the error term in the form $Kf^{IV}(c)$ (where $f^{IV}$ denotes the fourth derivative of $f$) for some value $c \in (-h, h)$ where $K$ is a constant to be determined. Show that the error in the integral, $I$, is consistent with this error term.

**4.** (a) Consider the ordinary differential equation,

$$\frac{dy}{dt} = f(t, y). \tag{1}$$

By integrating equation (1) over the interval $[t_n, t_{n+1}]$ and approximating $f(t, y)$ as a constant derive the forward Euler approximation,

$$y_{n+1} = y_n + hf(t_n, y_n),$$

where $h = t_{k+1} - t_k$ (for all $k$).

Verify that $y(t) = \cos(t)$ is the solution the initial value problem

$$\frac{dy}{dt} = \cos(t) - \sin(t) - y, \qquad y(0) = 1. \tag{2}$$

Use the forward Euler method to calculate an approximation to $y(\pi/3)$ using a step-size of $h = \pi/6$ and calculate the absolute error from the exact solution $y(\pi/3) = 0.5$.

(b) By finding a suitable quadrature formula for integration of equation (1) over the interval $[t_n, t_{n+1}]$ derive the three-step Adams-Bashforth scheme,

$$y_{n+1} = y_n + \frac{h}{12}\big(23f(t_n, y_n) - 16f(t_{n-1}, y_{n-1}) + 5f(t_{n-2}, y_{n-2})\big).$$

**5.** (a) Find the $LU$-factorisation of the matrix,

$$\begin{pmatrix} -2 & 3 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix},$$

showing your method of working. Demonstrate how this factorisation may be used to solve the linear system,

$$\begin{pmatrix} -2 & 3 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \\ 3 \end{pmatrix}.$$

(b) Explain what is meant by (row) *pivoting* in the context of matrix factorisation. Show that the linear system,

$$\begin{pmatrix} 0 & 2 & 1 \\ 2 & -2 & 4 \\ 1 & -1 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix}.$$

cannot be solved by $LU$-factorisation without pivoting even though the matrix is non-singular. Show how pivoting overcomes this problem and hence find the solution.

END