MATH260001

This question paper consists of 3 printed pages, each of which is identified by the reference **MATH260001**.

Only approved basic scientific calculators may be used.

Examination for the Module MATH2600
(January 2006)

Numerical Analysis

Time allowed: **2 hours**

Answer FOUR of the FIVE questions.

All questions carry equal marks.

1. (a) Show that the function $f(x) = x - \cos x$ has a unique zero in the interval $[0, \pi/2]$.

(b) Describe briefly how this solution may be found using the bisection method, and perform the first three iterations. How many iterations are needed to guarantee that the absolute error is less than 10^{-4}?

(c) Write down the general step of Newton's method for finding the solution $f(p) = 0$ in the form

$$p_{n+1} = g(p_n).$$

Show that $g'(p) = 0$ provided that $f'(p) \neq 0$ and hence show that this method has quadratic convergence.

Taking $p_0 = 1$ as an initial guess for the solution of $f(x) = x - \cos x$ calculate the next three iterates using Newton's method. You should work to 5 decimal places throughout and show all your working.

2. (a) Let $x_1 < x_2 < \cdots < x_n$. Write down an expression for the polynomial $q_k(x)$ of degree $n-1$ that satisfies the conditions,

$$q_k(x) = \begin{cases} 1, & x = x_k; \\ 0, & x = x_i \qquad i \neq k \end{cases}$$

Hence find the equation of the polynomial $P(x)$, of degree $n-1$, such that

$$P(x) = f(x) \qquad \text{for } x = x_1, x_2 \ldots x_n.$$

 QUESTION 2 CONTINUED...

(b) The value of the function $f(x)$ is known at the points $x = 0, 1$ and 2 and is given by

$$f(0) = 1, \qquad f(1) = 2, \qquad f(2) = -1. \tag{1}$$

Find the quadratic polynomial that interpolates $f(x)$ through these points and use it to approximate $f(0.5)$ and $f(1.5)$.

(c) A function g is defined on the interval [0,2] by,

$$g(x) = \begin{cases} g_1(x), & x \in [0,1), \\ g_2(x), & x \in [1,2], \end{cases}$$

where g_1 and g_2 are cubic polynomials. Write down the system of equations that must be satisfied if g is a natural cubic spline through the points in equation (1). If $g_1 = -x^3 + c_1x^2 + 2x + a_1$ and $g_2 = d_2(x-1)^3 + c_2(x-1)^2 + b_2(x-1) + a_2$, determine the constants a_1, c_1, a_2, b_2, c_2, d_2 so that g satifies these conditions. Hence determine $g(0.5)$ and $g(1.5)$.

3. (a) The trapezium rule for integrating a function over the interval $[0, h]$ can be written in the form,

$$\int_0^h f(x)dx = w_1 f(0) + w_2 f(h) + K_1 f''(c)$$

w_1 and w_2 are the weights and $K_1 f''(c)$ is the error term for some value $c \in [0, h]$. Use the method of undetermined coefficients to find the values of the constants w_1, w_2 and K_1. Calculate the trapezium rule approximation to the integral

$$I = \int_0^{\pi/2} \sin(x)dx,$$

and calculate the error from the actual answer. Show this is consistent with the error term.

(b) Show that the midpoint rule

$$\int_0^h f(x)dx = hf\left(\frac{h}{2}\right)$$

is also exact for all linear polynomials and find the error term in the form $K_2 f''(c)$. Calculate the approximation to the integral I using this method and the error from the exact answer.

(c) Obtain a higher order estimate for the integral I using a linear combination of your results from the trapezium and midpoint methods that eliminates the leading order error term.

4. (a) Write down the *modified Euler* method for solving the initial value problem,

$$\frac{dy}{dt} = f(t, y) \qquad \text{with } y(0) = y_0.$$

Verify that $y(t) = t/(1+t)$ is the solution of the initial value problem

$$\frac{dy}{dt} = (y-1)^2, \qquad y(0) = 0. \tag{2}$$

Use the modified Euler method to calculate an approximation to $y(1)$ using a step-size of $h = 0.5$ and calculate the absolute error from the exact solution $y(1) = 0.5$.

(b) Use the method of undetermined coefficients to find constants w_0, w_{-1} and w_{-2} for the quadrature formula,

$$\int_0^h g(t)dt = w_0 g(0) + w_{-1} g(-h) + w_{-2} g(-2h).$$

By using this quadrature formula to approximate

$$\int_{t_n}^{t_{n+1}} f(t,y)dt,$$

derive the three-step Adams-Bashforth scheme,

$$y_{n+1} = y_n + \frac{h}{12}\big(23f(t_n,y_n) - 16f(t_{n-1},y_{n-1}) + 5f(t_{n-2},y_{n-2})\big).$$

Find the approximation to the solution of equation (2) at $t = 0.5$ using a step size of $h = 0.5$ and using the exact solutions for $y(0.5)$ and $y(1.0)$.

5. (a) Explain what is meant by an LU factorisation of a square matrix $\mathbf{A}$.
Find the LU-factorisation of the matrix,

$$\begin{pmatrix} 4 & 4 & 1 \\ 2 & 1 & 1 \\ 2 & 1 & 2 \end{pmatrix},$$

showing your method of working. Demonstrate how this factorisation may be used to solve the linear system,

$$\begin{pmatrix} 4 & 4 & 1 \\ 2 & 1 & 1 \\ 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}.$$

(b) Explain what is meant by (row) *pivoting* in the context of matrix factorisation. Find the LU factorisation of the following matrix; (i) without pivoting; and (ii) with row pivoting,

$$\begin{pmatrix} 0.001 & 3 \\ 1 & 2 \end{pmatrix}.$$

Explain why pivoting provides more stable solutions when using floating point arithmetic.

END