MATH–259101

This question paper consists of 5 printed pages, each of which is identified by the reference MATH–259101

Only approved basic scientific calculators may be used.

Examination for the Module MATH–2591

(May/June 2005)

Dynamics of Particles and Rigid Bodies

Time allowed: 2 hours

Do not answer more than **FOUR** question.
All questions carry equal marks.

1. **(a)** Define angular momentum ($\mathbf{h}$) and the moment of force ($\mathbf{G}$) around a point O. Prove that $d\mathbf{h}/dt = \mathbf{G}$ and use this to show that motion under the central force of magnitude μ/r^2 directed towards point O, obeys conservation of angular momentum. Here μ is a positive constant and r the distance of the particle from O. Thus show that the path of a particle under this force lies entirely in a fixed plane through the point O.

(b) A particle of unit mass moves in this fixed plane under the action of the central force described in part (a). Using plane polar coordinates (r, θ) with non-constant unit vectors $\hat{\mathbf{r}} = r^{-1}\mathbf{r} = (\cos\theta, \sin\theta)$ and $\hat{\boldsymbol{\theta}} = (-\sin\theta, \cos\theta)$, show that $\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$ and $h = r^2\dot{\theta}$, the latter being conserved according to part (a). Here the notation $\dot{\mathbf{r}} = d\mathbf{r}/dt$ and $\dot{\theta} = d\theta/dt$ is used. Another conserved physical quantity is $\frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \mu/r$. Name this quantity.

(c) Since $h > 0$, θ is an increasing function of t and as such r may be expressed as a function of θ instead of t. Using $u = 1/r$ and the results in part (b), express the transverse velocity component in terms of h and u. Also, using the chain rule, show that the radial velocity component is equal to $-h du/d\theta$. Hence deduce that

$$\frac{h^2}{2}\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] - \mu u = E. \tag{1}$$

(d) Name the quantities $l = h^2/\mu \leq 0$ and $e = \sqrt{2Eh^2/\mu^2 + 1} \geq 0$ and show that equation (1) can be written as $(dw/d\theta)^2 = 1 - w^2$, where $w = (lu - 1)/e$. Using the substitution $w = \cos\psi$, show that this becomes $lu = 1 + e\cos(\theta - \theta_0)$.

(e) According to Kepler's first law, each planet describes an ellipse with the sun at one focus. Use the last expression in part (d) to derive expressions for the maximum and minimum distance from the sun. What are these positions called? Use them to calculate the semi-major axis of the elliptic orbit.

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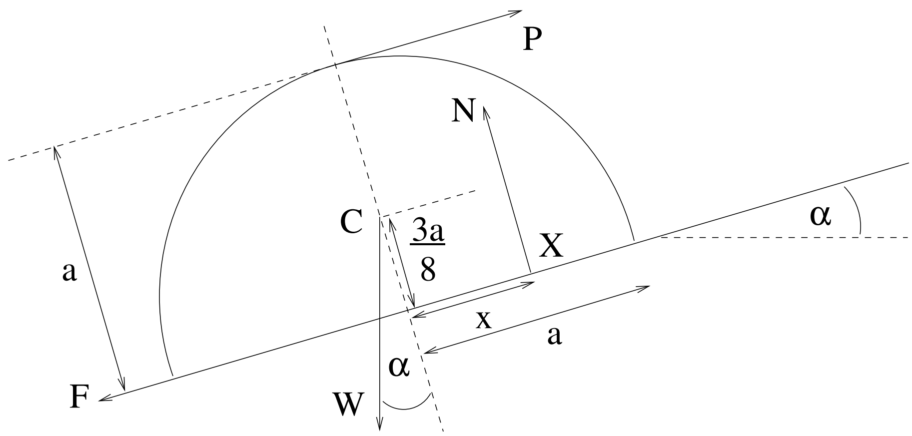
2. **(a)** Define the *center of mass* (CM) of a rigid body. Using this definition, show that the CM of a uniform solid of revolution is at position

$$\bar{z} = \frac{\int_{z_1}^{z_2} z r^2 dz}{\int_{z_1}^{z_2} r^2 dz}$$

using polar coordinates (r, θ, z) and with the z axis as axis of symmetry. Let the revolving body be given by $0 \le r \le r(z)$ with $z_1 \le z \le z_2$. Using this formulation, show that a solid hemisphere of radius a has $\bar{z} = 3a/8$, i.e. its centre of mass lies on its axis of symmetry at a distance $3a/8$ from its circular plane face, henceforth to be called the base of the body.

(b) The solid hemisphere described in part (a), whose weight is W, rests in equilibrium with its flat face in contact with an inclined plane that has a rough surface, giving the latter a coefficient of static friction μ. The plane has an inclination α with the horizontal, while the point of the body most remote from its base is subjected to a force of magnitude P, parallel to the line of slope of the plane in the upward direction. By resolving along and



normal to the incline, obtain expressions in terms of P, W and α for the downward frictional force F and the normal reaction N exerted by the plane on the body, and show that

$$\frac{F}{N} = \frac{P}{W}\sec\alpha - \tan\alpha.$$

(c) The line of action of N meets the diameter of the base of the body at a point at a distance x up the incline from the centre of the base. Take moments about the centre of the base and use the expression previously obtained for N, to show that

$$\frac{x}{a}\cos\alpha = \frac{P}{W} - \frac{3}{8}\sin\alpha.$$

(d) Given that $\tan\alpha < \mu$, use the law of static friction to show that $P \le W(\sin\alpha + \mu\cos\alpha)$. Also, obtain from part (c) the result $P \le W(8\cos\alpha + 3\sin\alpha)/8$.

(e) P is now gradually increased, while α is held fixed. Show that equilibrium is eventually broken by slipping when $P = W(\sin\alpha + \mu\cos\alpha)$ or by toppling when $P = W(8\cos\alpha + 3\sin\alpha)/8$, according as μ is less or greater than $(8 - 5\tan\alpha)/8$.

3. **(a)** For a system of forces $\mathbf{F}_\alpha$ acting at points $\mathbf{r}_\alpha$, define the *resultant* $\mathbf{F}$ and the *resultant moment* $\mathbf{G}$ about the origin. Show that for the resultant moment $\mathbf{G_r}$ about an arbitrary point $\mathbf{r}$, we can write $\mathbf{G_r} = \mathbf{G} - \mathbf{r} \times \mathbf{F}$, and that this satisfies $\mathbf{F}{\bullet}\mathbf{G_r} = \mathbf{F}{\bullet}\mathbf{G}$ independent of $\mathbf{r}$.

(b) Define briefly what is meant by a *couple* and its particular *moment*, as well as a *wrench* of a particular *intensity* and *pitch*. Using part (a), derive the parametric expression for the central axis of a wrench. Show that the pitch can be written as a function of invariants and state without proof necessary and sufficient conditions on resultant $\mathbf{F}$ and resultant moment $\mathbf{G}$ for the system in part (a) to be equivalent to (i) a wrench, (ii) a single force, (iii) a couple, or (iv) no forces at all.
[Accept without proof the vector identity $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A}{\bullet}\mathbf{C})\mathbf{B} - (\mathbf{A}{\bullet}\mathbf{B})\mathbf{C}$.]

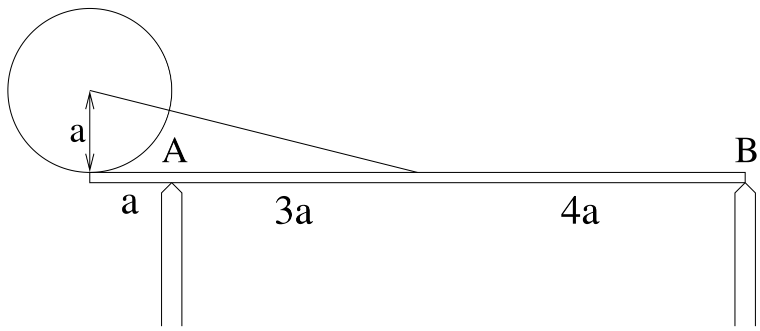
(c) Forces $\mathbf{F}_1 = (2, 1, -1)$ and $\mathbf{F}_2 = (-2\beta, 2 - 3\beta, \beta)$ act at the points $\mathbf{r}_1 = (\beta, 1, 2\beta)$ and $\mathbf{r}_2 = (1, 2, 1)$ respectively. Show that the system has resultant $\mathbf{F} = (1 - \beta)(2, 3, -1)$ and resultant moment $\mathbf{G} = [3(\beta - 1), 2\beta, 2\beta]$, and calculate $\mathbf{F}{\bullet}\mathbf{G}$.

(d) Find the unique value of β $(= \beta_c)$ for which the system in part (c) is equivalent to a couple, and write down the components of the moment of that couple.

(e) Find the unique value of β $(= \beta_f)$ for which the system in part (c) is equivalent to a single force. Write down the components of that force and obtain a parametric equation for its line of action.

(f) Show that for all values of β other than β_f and β_c, the system in part (c) is equivalent to a wrench. Calculate the intensity and pitch of this wrench as a function of β.

4. A car park barrier consists of a uniform thin pole of length $8a$ and mass m with a uniform circular disc of radius a and mass $2m$ welded tangentially at one end. A light tie joins the centre of the disc with the centre of the pole (for added strength). Assume that this tie is weightless. The barrier is supported on the left by a rough bracket A a distance a from the end of the pole, while the right end of the pole rests on a support B.



(a) Use the principle of decomposition to show that the centre of mass of the barrier lies on the tie, a distance $a/3$ to the right of the bracket A which supports the barrier and $2a/3$ above the horizontal line AB. (Use A as the origin of the reference system.)

(b) Calculate the moment of inertia (MI) of the pole about A. Assuming (without proof) that the MI of a uniform solid disc with mass M and radius R is $MR^2/2$, use the parallel axes theorem to determine the MI of the uniform circular disc of the barrier about the point A. Show that the total MI of the barrier is $58ma^2/3$

(c) The barrier is designed so that it pivots about A in a vertical plane, and has an equilibrium position that is partly open. Draw a diagram showing all the forces acting on the barrier when the pole makes an angle θ with the horizontal line AB. Show that the angle $\tan\theta = 1/2$ when the barrier is in its equilibrium position.

(d) By working in polar coordinates (r, θ) and taking A to be the origin of the system, show that the forces along and perpendicular to the pole satisfy the relations

$$F = m\left[3g\sin\theta - \sqrt{5}\,a\dot{\theta}^2\right] \quad \text{and} \quad N = m\left[3g\cos\theta + \sqrt{5}\,a\ddot{\theta}\right]$$

by applying Newton's second law, where F is the frictional force directed upward along the pole and N the normal reaction of bracket A on the pole.
[It may be assumed without proof that the radial and transverse acceleration in polar coordinates are given by $(\ddot{r} - r\dot{\theta}^2)$ and $(r\ddot{\theta} + 2\dot{r}\dot{\theta})$ respectively.]

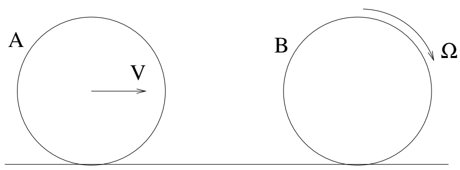
(e) The barrier is disturbed from its equilibrium position and falls towards the horizontal. Write down the expression for the total energy and show that at equilibrium this is equal to $5mga\sin\theta$. [Take A as the reference level for the potential energy.] Use the principle of energy conservation to show that the angular velocity of the barrier is given by

$$\dot{\theta}^2 = \frac{6}{29}\frac{g}{a}\left(2\sin\theta - \cos\theta\right).$$

Show by differentiating with respect to t that

$$\ddot{\theta} = \frac{3}{29}\frac{g}{a}\left(2\cos\theta + \sin\theta\right).$$

5. A hollow uniform tube A and a hollow uniform sphere B, each of mass M and radius a, lie (not in contact with each other) on a fixed rough horizontal plane, with which each has the same coefficient of dynamic friction μ'. At time $t = 0$ tube A has zero angular velocity, but its centre is moving directly towards B with speed V. At the same time sphere B is instantaneously at rest, but has an angular velocity Ω about the horizontal diameter perpendicular to the line of centres in the sense required to make B roll away from A.



(a) Use the definition for the moment of inertia (MI) to show that the MI of the hollow tube A about its centre of mass is Ma^2. Assume (without proof) that the MI of the hollow sphere B about its centre of mass is given by $2Ma^2/3$.

(b) Show that

1. while tube A is slipping, it has uniform deceleration $\mu' g$ at the centre together with uniform angular acceleration $\mu' g/a$;
2. while sphere B is slipping, it has uniform acceleration $\mu' g$ at the centre together with uniform angular deceleration $3\mu' g/(2a)$.

(c) Show that, if $\Omega = 5V/(4a)$, then the tube and the sphere start to roll simultaneously at time $t = V/(2\mu' g)$.

(d) Show that when they start rolling, they both move at the same speed of $V/2$. One specification of the system is that tube A and sphere B should never touch each other. Show that during the time interval $0 \leq t \leq V/(2\mu' g)$, tube A moves a distance $3V^2/(8\mu' g)$ towards B while sphere B moves distance $V^2/(8\mu' g)$ away from A. Hence write down the minimum distance they should be apart at time $t = 0$ to satisfy the no-contact specification.

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