MATH239001

This question paper consists of 2 printed pages, each of which is identified by the reference MATH239001

Only approved basic scientific calculators may be used.

Examination for the Module MATH2390

(May/June 2004)

Dynamical Systems

Time allowed: 2 hours

Answer **four** questions.
All questions carry equal marks.

1. **(a)** For each of the following one-dimensional dynamical systems $\frac{dy}{dt} = f(y)$, plot the vector field $f(y)$ as a function of y, identify equilibrium points and their stability, and determine the behaviour of solutions, plotting $y(t)$ as a function of t for different choices of initial condition $y(0)$.

(i) $\dfrac{dy}{dt} = 3y - y^2$ **(ii)** $\dfrac{dy}{dt} = (y^2-1)(y-2)^2$ **(iii)** $\dfrac{dy}{dt} = -y + 4\cos y$

(b) Sketch the bifurcation diagrams for the following one-dimensional dynamical systems $\frac{dy}{dt} = f(y, \mu)$, obtaining a plot of the equilibrium points against μ and indicating their stability. Identify the type of each bifurcation your figure.

(i) $\dfrac{dy}{dt} = -\mu y - y^3$ **(ii)** $\dfrac{dy}{dt} = y(\mu - 2y + y^2)$ **(iii)** $\dfrac{dy}{dt} = \mu + 3y - y^3$

2. **(a)** For the two-dimensional linear dynamical system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix},$$

where $\mathbf{A}$ is a matrix, use $T = \text{trace}(\mathbf{A})$ and $D = \det(\mathbf{A})$ to classify the equilibrium point $(0,0)$, indicating in the (T, D) plane where each different type of phase portrait can be found.

(b) Sketch the vector field and phase portrait for each of the following two-dimensional dynamical systems, and state whether or not the equilibrium point $(0,0)$ is hyperbolic.

(i) $\dfrac{dx}{dt} = -3x - 2y, \quad \dfrac{dy}{dt} = 2x + 2y.$

(ii) $\dfrac{dx}{dt} = -3x - 4y, \quad \dfrac{dy}{dt} = 2x + y.$

(iii) $\dfrac{dx}{dt} = -x - 3y, \quad \dfrac{dy}{dt} = x + 3y.$

continued ...

3. **(a)** Suppose that $P(x, y)$ and $Q(x, y)$ are continuous functions of x and y in a neighbourhood of $(x, y) = (0, 0)$, with $P(0, 0) < 0$ and $Q(0, 0) < 0$. Show that the equilibrium point $(0, 0)$ is stable for the following second-order differential equation:

$$\dot{x} = xP(x, y), \qquad \dot{y} = yQ(x, y).$$

(b) For which values of k is $V(x, y) = x^2 + ky^2$ a Lyapunov function for the following second-order differential equation:

$$\dot{x} = -x + y + y^2, \qquad \dot{y} = x - 2y.$$

By choosing $k = 1$, show that all trajectories that start with $x^2 + y^2 < 1$ are attracted to the equilibrium point $(0, 0)$.

4. **(a)** State the Poincaré Index for a stable node, a saddle point and a periodic orbit.

(b) Sketch the phase portrait of the Lotka–Volterra system:

$$\dot{x} = x(4 - x - 3y), \qquad \dot{y} = y(3 - 2x - y).$$

(c) Using either Poincaré Index theory, or Dulac's Criterion, explain carefully why the previous set of equations has no periodic solution in the positive quadrant.

5. **(a)** Find the equilibrium point of the following second-order set of differential equations:

$$\dot{x} = a - (b + 1)x + x^2 y, \qquad \dot{y} = bx - x^2 y,$$

where a and b are positive parameters. Show that there is a Hopf bifurcation when $b = 1 + a^2$.

(b) State the Poincaré–Bendixson Theorem. Use the theorem to show that the following second-order set of differential equations has a periodic solution:

$$\dot{x} = x - y - \left(x^2 + \frac{1}{2}y^2\right) x, \qquad \dot{y} = x + y - \left(x^2 + \frac{3}{2}y^2\right) y.$$

END