MATH–221001

This question paper consists of 3 printed pages, each of which is identified by the reference MATH–2210

Only approved basic scientific calculators may be used.

Examination for the Module MATH–2210
(May 2006)

Introduction to Discrete Mathematics
Time allowed : 2 hours

Answer not more than **four** questions. All questions carry equal marks.

1. (a) In the National Lottery of Bolonia, participants select 8 numbers between 1 and 80 inclusive. 12 numbers in the same range are then drawn at random. You win a first prize if all your numbers are among the 12 drawn, and you win a consolation prize if all but one of your numbers are among the 12 drawn. What is the probability that you win a first prize? What is the probability that you win a second prize?

(b) For finite sets A_1 and A_2, prove that

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

State the inclusion-exclusion principle for n finite sets $A_1, A_2, \ldots, A_n$ in terms of numbers of the form $|A_{r_1} \cap A_{r_2} \cap \ldots \cap A_{r_k}|$ for $1 \leq r_1 < r_2 < \ldots < r_k \leq n$.

Find how many integers there are in the range 1 to 1000 which are perfect squares, or multiples of either 5 or 7.

2. (a) Find the general solution of the difference equation

$$x_{n+2} - 5x_{n+1} + 4x_n = 2^n.$$

(b) Find the solution of the difference equation

$$x_{n+1} - 3x_n = 3^n$$

such that $x_0 = 3$.

(c) Let y_n stand for the number of strings of length n in the digits 0, 1, 2, …, 9 in which 3 occurs an odd number of times. Find a (first order) difference equation for y_n, and hence find a formula for y_n in terms of n. Check your formula for the cases $n = 1$ and 2.

3. (a) State and prove the *Handshaking Lemma*.

(b) In each of the following cases, where the degrees of the vertices of a (simple) graph with 5 vertices are specified, either draw a picture of a graph which matches the given information, or prove that no such graph exists.

G_1

vertex	degree
a	3
b	4
c	3
d	4
e	3

G_2

vertex	degree
a	4
b	3
c	1
d	1
e	1

G_3

vertex	degree
a	4
b	3
c	3
d	3
e	3

(c) (i) Define what is meant by saying that a graph is a *tree*.

(ii) List all non-isomorphic trees with at most 5 vertices.

(iii) Prove that for all $v \geq 1$, a tree with v vertices has $v - 1$ edges.

4. (a) Let G be a weighted simple graph, where the weights (on edges) are positive real numbers. Explain what a *minimal connector* for G is, and prove that this exists (provided G is finite and connected).

(b) Find a minimal connector for the weighted graph shown using Kruskal's algorithm, and give the value of the minimum total weight.

edge	μ	edge	μ
ab	13	cg	10
ae	14	ch	11
af	32	de	25
ah	20	dg	28
bc	18	dh	39
be	8	ef	2
bf	5	fg	23
cd	37	gh	4

(c) State without proof Euler's formula for finite connected graphs drawn in the plane having v vertices, e edges, and f faces, and deduce that $e \leq 3v - 6$ provided that there are at least 3 vertices.

(d) Draw the complete graph K_5 on 5 vertices, and deduce from part (c) that it is not planar.

5. (a) Consider the following register machine program P:

$\hat{1}$ (1, 2, 4)	$\hat{5}$ (1, 4)
$\hat{2}$ (2, 3)	$\hat{6}$ (1, 7)
$\hat{3}$ (1, 1, 6)	$\hat{7}$ Halt
$\hat{4}$ (2, 5, 7)	

(i) Draw the flow chart corresponding to P.

(ii) Give the full trace table of the computation for the single integer inputs 3 and 4 (where register 2 is as usual assumed initialized to 0)

(iii) Describe the function $f : \mathbb{N} \to \mathbb{N}$ computed by P.

(b) Modify the program P given in part (a) to compute the function $g(n) = \begin{cases} 5 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$

6. (a) Define what is meant by saying that a function f of n variables is *primitive recursive.*

Prove that addition $A(x, y) = x + y$ and multiplication $M(x, y) = xy$ are both primitive recursive.

(b) By exhibiting suitable programs, show that each of the basic functions $Z(x)$, $S(x)$, and $U_i^n(x_1, \ldots, x_n)$ is register machine computable, and show that if $f(x)$ and $g(x)$ are register machine computable, then so is $f(x) + g(x)$.

(c) Show that there is a non-computable function $f : \mathbb{N} \to \mathbb{N}$.

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