MATH-209001

Only approved basic scientific calculators may be used.

This question paper consists of 2 printed pages, each of which is identified by the reference MATH-209001.

Examination for the Module MATH-2090
(May–June 2007)

Real and Complex Analysis

Time allowed : 2 hours

Do not answer more than **four** questions
All questions carry equal marks

1. **(a)** Show that the improper integral

$$\int_3^\infty \frac{1}{(x+2)^2}\,\mathrm{d}x$$

converges, and find its value.

(b) Give the "ε–δ" definition of continuity for a function $f : \mathbb{R} \to \mathbb{R}$ at a point $a \in \mathbb{R}$, and use it to show that the function f defined by $f(x) = 5x + 1$ is continuous at every point $a \in \mathbb{R}$.

(c) What is meant by saying that a function f is *differentiable* at the point $a \in \mathbb{R}$? Show from the definition that the function f defined by $f(x) = 1/(x + 2)$ (for $x \neq -2$) is differentiable at the point $a = 4$, and calculate its derivative there.

(d) State the *Intermediate Value Theorem* and the *Mean Value Theorem.*

Use one of the above results to show that the equation $x^4 = e^x$ has a solution in the interval $[-1, 0]$ and another solution in the interval $[1, 2]$.

2. **(a)** State the *Cauchy–Riemann equations* for a function $f(z) = u(x, y) + \mathrm{i}v(x, y)$, where $z = x + \mathrm{i}y$.

For each of the following functions f determine the corresponding u and v, and find the values of z for which the Cauchy–Riemann equations are satisfied.

(i) $f(z) = \overline{z}^2 + \mathrm{i}\,\mathrm{Re}\, z\,;$ **(ii)** $f(z) = \mathrm{i}z^2 - 3z.$

Question 2 continues . . .

(b) Define the notion of a *harmonic* function. Suppose that $u(x, y)$ and $v(x, y)$ are smooth functions satisfying the Cauchy–Riemann equations. Prove that $u(x, y)$ is harmonic.

Show that the function $u(x, y) = e^x \sin y + x - y$ is harmonic, and find a harmonic conjugate function.

3. **(a)** State what it means for a subset $U \subseteq \mathbb{C}$ to be *open*. State what it means for U to be *path-connected*.

(b) Sketch each of the following sets, and say (with brief reasons) which of them are open and which are path-connected.

(i) $\{z \in \mathbb{C} : |z| \neq 2\}$; **(ii)** $\{z \in \mathbb{C} : z - \overline{z} = 2\mathrm{i}\}$.

(c) Let $p : [a, b] \to \mathbb{C}$ define a piecewise smooth path, and let f be a continuous function defined on the path p. Give the definition of the expression $\displaystyle\int_p f(z)\,\mathrm{d}z$.

(d) Let $p : [0, 2\pi] \to \mathbb{C}$ be the path defined by $p(t) = 4e^{\mathrm{i}t}$. Evaluate the following integrals, stating which standard results you use:

(i) $\displaystyle\int_p \frac{1}{z^5}\,\mathrm{d}z$; **(ii)** $\displaystyle\int_p \overline{z}\,\mathrm{d}z$; **(iii)** $\displaystyle\int_p \frac{z^3}{z-1}\,\mathrm{d}z$; **(iv)** $\displaystyle\int_p \cos(e^z)\,\mathrm{d}z$.

4. **(a)** Find the Taylor series of the function e^z about the point $z = 2$, and state the radius of convergence of the power series.

(b) **(i)** Compute the Laurent series of the function

$$f(z) = \frac{1}{z(z-3)}$$

about the point $z = 3$, valid in the region $\{z \in \mathbb{C} : 0 < |z - 3| < 3\}$.

(ii) Compute the Laurent series of the same function, about the same point, valid in the region $\{z \in \mathbb{C} : 3 < |z - 3|\}$.

(c) What does it mean to say that a function f has a *pole of order N at z_0*? Give the definition of the *residue* of f at such a pole.

For each of the following functions, determine whether it has a pole at $z = 0$, and, if it does, determine the order of the pole and the residue at 0:

(i) $\displaystyle\frac{\sin z}{z}$; **(ii)** $\displaystyle\frac{\cos z}{z}$; **(iii)** $\displaystyle\frac{1}{z^2(z+1)}$.

5. Use complex variable methods to evaluate:

(i) $\displaystyle\int_0^{2\pi} \frac{1}{3 + 2\cos t}\,\mathrm{d}t$; **(ii)** $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^2 - 2x + 2}\,\mathrm{d}x$.

END