MATH-209001

Only approved basic scientific calculators may be used.

This question paper consists of 2 printed pages, each of which is identified by the reference MATH-209001.

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Examination for the Module MATH-2090

(May–June 2006)

Real and Complex Analysis

Time allowed : 2 hours

Do not answer more than **four** questions

All questions carry equal marks

1. **(a)** Show that the improper integral

$$\int_2^\infty \frac{1}{x(x+7)}\,dx$$

converges, and find its value.

(b) Give the "ε–δ" definition of continuity for a function $f : \mathbb{R} \to \mathbb{R}$ at a point $a \in \mathbb{R}$, and use it to show that the function f defined by $f(x) = 1 + 2|x|$ is continuous at the point $x = 0$.

(c) What is meant by saying that a function f is *differentiable* at the point $a \in \mathbb{R}$? Explain briefly why the function $f(x) = 1 + 2|x|$ is not differentiable at $x = 0$.

(d) State *Rolle's Theorem.*

By considering the function $x \mapsto (x+1)\sin x$, or otherwise, show that there is a solution to the equation

$$(x+1)\cos x + \sin x = 0$$

in the interval $[0, \pi]$.

2. **(a)** For each of the following functions $f(z)$, find real-valued functions u and v such that $f(z) = u(x,y) + iv(x,y)$, where $z = x + iy$.

$$\text{(i) } f(z) = iz^3 + 2z\,; \qquad \text{(ii) } f(z) = |z|^2 - \overline{z}.$$

(b) State the *Cauchy–Riemann equations.* For each of the two functions in Part (a), state precisely the values of z for which the Cauchy–Riemann equations are satisfied.

Question 2 continues . . .

(c) Define the notion of a *harmonic* function. Suppose that $u(x, y)$ and $v(x, y)$ are smooth functions satisfying the Cauchy–Riemann equations. Prove that $v(x, y)$ is harmonic.

For each of the functions in Part (a), determine whether or not the functions $u(x, y)$ and $v(x, y)$ are harmonic.

3. **(a)** State what it means for a subset $U \subseteq \mathbb{C}$ to be *open*. State what it means for U to be *path-connected*.

(b) Sketch each of the following sets, and say (with brief reasons) which of them are open and which are path-connected.

$$\text{(i) } \{z \in \mathbb{C} : z + \overline{z} \neq 1\}; \qquad \text{(ii) } \{z \in \mathbb{C} : z\overline{z} = 4\}.$$

(c) Let $p : [a, b] \to \mathbb{C}$ define a piecewise smooth path, and let f be a continuous function defined on the path p. Give the definition of the expression $\displaystyle\int_p f(z)\, dz$.

(d) State *Cauchy's integral formula.*

(e) Let $p : [0, 2\pi] \to \mathbb{C}$ be the path defined by $p(t) = 2 + e^{it}$. Sketch this path, and evaluate the following integrals:

$$\text{(i) } \int_p \frac{e^z}{z}\, dz\,; \qquad \text{(ii) } \int_p |z|^2\, dz\,; \qquad \text{(iii) } \int_p \frac{z^4}{z-2}\, dz.$$

4. **(a)** Find the Taylor series of the function $1/(z + 1)$ about the point $z = 2$, and state the radius of convergence of the power series.

(b) **(i)** Compute the Laurent series of the function

$$f(z) = \frac{1}{(z+1)(z-2)}$$

about the point $z = 2$, valid in the region $\{z \in \mathbb{C} : 0 < |z - 2| < 3\}$.

(ii) Compute the Laurent series of the same function, about the same point, valid in the region $\{z \in \mathbb{C} : 3 < |z - 2|\}$.

(c) What does it mean to say that a function f has a *pole of order N at z_0*? Give the definition of the *residue* of f at such a pole.

For the function f defined by

$$f(z) = \frac{1}{z^2(z^2-1)},$$

find the poles of f, and calculate the residue at each pole.

5. Use complex variable methods to evaluate:

$$\text{(i) } \int_0^{2\pi} \cos^6 t\, dt\,; \qquad \text{(ii) } \int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)}\, dx.$$

END