MATH-209001

Only approved basic scientific calculators may be used.

This question paper consists of 2 printed pages, each of which is identified by the reference MATH-209001.

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Examination for the Module MATH-2090

(May/June 2005)

Real and Complex Analysis

Time allowed : 2 hours

Do not answer more than **four** questions
All questions carry equal marks

1. **(a)** For each of the following improper integrals, find whether the integral converges and, if it does, find its value:

$$\text{(i)} \int_0^\infty e^{-3x}\,dx; \qquad \text{(ii)} \int_0^\infty \cos x\,dx.$$

(b) Define what is meant by saying that a function $f(x)$ is *continuous* at $x = a$. By using appropriate rules for continuous functions, and assuming that the functions $x \mapsto e^x$, $x \mapsto x$ and constant functions are continuous, show that the function

$$f(x) = \frac{e^x}{x^4 + 1}$$

is continuous at all $a \in \mathbb{R}$.

(c) State the *Intermediate Value Theorem.*

Show that there is some value of x in the range $0 < x < 2$ with

$$\frac{e^x}{x^4 + 1} = \frac{1}{2}.$$

2. **(a)** State the *Cauchy–Riemann equations* for a function $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$.

For each of the following functions f determine the corresponding u and v, and find the values of z for which the Cauchy–Riemann equations are satisfied.

$$\text{(i) } f(z) = z|z|^2; \qquad \text{(ii) } f(z) = z^2 + iz.$$

Question 2 continues . . .

(b) Define the notion of a *harmonic* function. Prove that if f is an analytic function on an open set U such that u and v have continuous second-order partial derivatives, then u is a harmonic function.

Show that the function $u(x, y) = x^2 + 4xy - y^2$ is harmonic, and find a harmonic conjugate function.

3. **(a)** State what it means for a subset $U \subseteq \mathbb{C}$ to be *open*. State what it means for U to be *path-connected*.

(b) Sketch each of the following sets, and say (with brief reasons) which of them are open and which are path-connected.

$$\text{(i) } \{z \in \mathbb{C} : |z| \geq 2\}; \qquad \text{(ii) } \{z \in \mathbb{C} : \operatorname{Re} z + \operatorname{Im} z \neq 0\}.$$

(c) State and prove the *Fundamental Theorem of Path Integrals.*

(d) Let $p : [0, 2\pi] \to \mathbb{C}$ be the path defined by $p(t) = 2 + e^{it}$. Evaluate the following integrals:

$$\text{(i) } \int_p \overline{z}\, dz; \qquad \text{(ii) } \int_p \frac{\cos z}{z-2}\, dz; \qquad \text{(iii) } \int_p \frac{e^z}{z+2}\, dz.$$

4. **(a)** Find the Taylor series of the function e^z about the point $z = 3$, and state the radius of convergence of the power series.

(b) Let $f(z)$ be the function defined by the formula

$$f(z) = \frac{1}{z(z-2)}.$$

(i) Find the Laurent series of the function $f(z)$ about the point $z = 2$, valid in the region $\{z \in \mathbb{C} : 0 < |z-2| < 2\}$;

(ii) Find the Laurent series of the same function, about the same point, valid in the region $\{z \in \mathbb{C} : 2 < |z-2|\}$.

(c) For each of the following functions, state the order of the pole at the point $z = 3$, and find the residue at that point:

$$\text{(i) } \frac{1}{(z-3)(z-6)}; \qquad \text{(ii) } \frac{e^z}{z^2-9}; \qquad \text{(iii) } \frac{z^2}{(z-3)^3}.$$

5. Use complex variable methods to evaluate:

$$\text{(i) } \int_0^{2\pi} \frac{1}{5 + 3\cos t}\, dt; \qquad \text{(ii) } \int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\, dx.$$

END