MATH-209001

Only approved basic scientific calculators may be used.

This question paper consists of 2 printed pages, each of which is identified by the reference MATH-209001.

Examination for the Module MATH-2090
(May/June 2004)

Real and Complex Analysis

Time allowed : 2 hours

Do not answer more than **four** questions
All questions carry equal marks

1. **(a)** For each of the following improper integrals, find whether the integral converges and, if it does, find its value:

$$\text{(i)} \int_0^\infty \frac{1}{(x+4)^2}\,dx; \qquad \text{(ii)} \int_0^\infty \frac{1}{\sqrt{x+4}}\,dx.$$

(b) Define what is meant by saying that a function $f(x)$ is *continuous* at $x = a$. By using appropriate rules for continuous functions, and assuming that the functions $x \mapsto \cos x$, $x \mapsto x$ and constant functions are continuous, show that the function

$$f(x) = \frac{x^2}{\cos x + 2}$$

is continuous at all $x \in \mathbb{R}$.

(c) Show that there is some value of x in the range $0 < x < \pi$ with

$$\frac{x^2}{\cos x + 2} = \frac{\pi^2}{2}.$$

(d) State *Rolle's Theorem.*

2. **(a)** For each of the following functions $f(z)$, find real-valued functions u and v such that $f(z) = u(x,y) + iv(x,y)$, where $z = x + iy$.

$$\text{(i)}\ f(z) = z^3 + 2z; \qquad \text{(ii)}\ f(z) = |z|^2 + \overline{z}^2.$$

(b) State the *Cauchy-Riemann equations.* For each of the functions in (i) and (ii), state precisely the values of z for which the Cauchy-Riemann equations are satisfied.

(c) Define the notion of a *harmonic* function. For each of the functions in (i) and (ii), determine whether or not the functions $u(x,y)$ and $v(x,y)$ are harmonic.

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3. **(a)** State what it means for a subset $U \subseteq \mathbb{C}$ to be *open*. State what it means for U to be *path-connected.*

(b) Sketch each of the following sets, and say (with brief reasons) which of them are open and which are path-connected.

$$\text{(i) } \{z \in \mathbb{C} : 1 < |z| \leq 2\}; \qquad \text{(ii) } \{z \in \mathbb{C} : 1 < (\Re e z)^2 < 2\}.$$

(c) State *Cauchy's Theorem.*

(d) Let $p : [0, 2\pi] \to \mathbb{C}$ be the path defined by $p(t) = 3e^{it}$. Evaluate the following integrals:

$$\text{(i) } \int_p |z|\,dz; \qquad \text{(ii) } \int_p \frac{e^z}{z}\,dz; \qquad \text{(iii) } \int_p \frac{\sin z}{z+5}\,dz.$$

4. **(a)** Find the Taylor series of the function $1/z$ about the point $z = 1$, and state the radius of convergence of the power series.

(b) **(i)** Compute the Laurent series of the function $1/(z^2 - 1)$, about the point $z = 1$, valid in the region $\{z \in \mathbb{C} : 0 < |z - 1| < 2\}$;

(ii) Compute the Laurent series of the same function, about the same point, valid in the region $\{z \in \mathbb{C} : 2 < |z - 1|\}$.

(c) For each of the following functions, state the order of the pole at the point $z = 1$, and find the residue at that point:

$$\text{(i) } \frac{1}{z^2 - 1}; \qquad \text{(ii) } \frac{e^z}{(z-1)^2}; \qquad \text{(iii) } \frac{z^3}{(z-1)^3}.$$

5. Use complex variable methods to evaluate:

$$\text{(i) } \int_0^{2\pi} \frac{1}{3 + \cos t}\,dt; \qquad \text{(ii) } \int_{-\infty}^{\infty} \frac{1}{(x^2+4)^2}\,dx.$$

END