MATH-209001

Only approved basic scientific calculators may be used.

This question paper consists of 2 printed pages, each of which is identified by the reference MATH-209001.



Examination for the Module MATH-2090
(May/June 2003)

**Real and Complex Analysis**

Time allowed : 2 hours

Do not answer more than **four** questions
All questions carry equal marks

**A1.** **(a)** Show that the improper integral

$$\int_0^\infty \frac{1}{(x+1)(x+3)}\,dx$$

converges, and find its value.

**(b)** Define what is meant by saying that a function $f(x)$ is *continuous* at $x = a$, and prove directly from the definition that the function $g(x) = 3 + 2x$ is continuous at any point.

**(c)** By using appropriate rules for continuous functions, and assuming that the function $x \mapsto e^x$ is continuous, show that the function

$$f(x) = \frac{e^{3+2x}}{3+2x}$$

is continuous on the closed interval $[0, 1]$. State carefully the rules that you have used.

**(d)** State the *Mean Value Theorem.*

**A2.** **(a)** State the *Cauchy-Riemann equations* for a function $f(z)$, and show that they hold at any point where $f(z)$ is (complex) differentiable.

**(b)** Determine the points of the complex plane at which the function

$$z \mapsto (\Re e(z))^2 + \Im m(z)\, i$$

is (complex) differentiable.

**(c)** Define the notion of a *harmonic* function. For each of the following functions, find whether the function is harmonic, and if so find a conjugate function.

$$\text{(a) } u(x,y) = x^2 + xy - y^2; \qquad \text{(b) } w(x,y) = x^3 + y^2.$$

 **Question A2 continues . . .**

**A3.** **(a)** State what it means for a subset $U \subseteq \mathbb{C}$ to be *open*. State what it means for $U$ to be *(path)-connected.*

**(b)** Sketch each of the following sets, and say (with brief reasons) which of them are open and which are (path)-connected.

(a) $\{z \in \mathbb{C} : |z| < 1 \text{ or } |z| > 2\}$; (b) $\{z \in \mathbb{C} : \Re e(z) \geq \Im m(z)\}$.

**(c)** State *Cauchy's Theorem.*

**(d)** Let $p : [0, 2\pi] \to \mathbb{C}$ be the path defined by $p(t) = 2e^{it}$. Evaluate the following integrals:

$$\text{(a)} \int_p \overline{z}^2 \, dz; \qquad \text{(b)} \int_p \frac{\cos z}{z} \, dz; \qquad \text{(c)} \int_p \frac{\cos(z+3)}{z+3} \, dz.$$

**A4.** **(a)** Find the Taylor series of the function $\sin z$ about the point $z = 2$, and state its radius of convergence.

**(b)** Let $f(z)$ be the function defined by the formula

$$f(z) = \frac{1}{(z-1)(z-2)^2}.$$

**(i)** Find the Laurent series of $f(z)$ about the point $z = 1$ which is valid in the region $\{z \in \mathbb{C} : |z-1| > 1\}$.

**(ii)** Find the Laurent series of $f(z)$ about the point $z = 2$ which is valid in the region $\{z \in \mathbb{C} : 0 < |z-2| < 1\}$.

**(c)** For each of the following functions, state the order of the pole at the point $z = 2$, and find the residue at that point:

$$\text{(a)} \frac{1}{(z-1)(z-2)^2}; \qquad \text{(b)} \frac{e^z}{z(z-2)}; \qquad \text{(c)} \frac{\sin z}{(z-2)^3}.$$

**A5.** Use complex variable methods to evaluate:

$$\text{(i)} \int_0^{2\pi} \frac{1}{3+\sin t} \, dt; \qquad \text{(ii)} \int_{-\infty}^{\infty} \frac{1}{(x^2+9)^2} \, dx.$$

**END**