MATH-209001

Only approved basic scientific calculators may be used.

This question paper consists of 2 printed pages, each of which is identified by the reference MATH-209001.

Examination for the Module MATH-2090
(May/June 2002)

Real and Complex Analysis

Time allowed : 2 hours

Do not answer more than **four** questions
All questions carry equal marks

A1. **(a)** For each of the following improper integrals, find whether the integral converges and, if it does, find its value:

$$\text{(a)} \int_2^\infty \frac{1}{(x-1)^2}\,dx; \qquad \text{(b)} \int_2^\infty \frac{1}{\sqrt{x-1}}\,dx.$$

(b) Define what is meant by saying that a function $f(x)$ is *continuous* at $x = a$. By using appropriate rules for continuous functions, and assuming that the functions $x \mapsto e^x$, $x \mapsto x$ and constant functions are continuous, show that the function

$$f(x) = \frac{x^2}{e^x + 1}$$

is continuous at all $x \in \mathbb{R}$.

(c) Show that there is some value of x in the range $0 < x < 1$ with

$$\frac{x^2}{e^x + 1} = \frac{1}{2e}.$$

(d) State *Rolle's Theorem.*

A2. **(a)** State the *Cauchy-Riemann equations* for a function $f(z)$, and show that they hold at any point where $f(z)$ is (complex) differentiable.

(b) Determine the points of the complex plane at which the function $z \mapsto (\Re e(z))^2$ is (complex) differentiable.

(c) Define the notion of a *harmonic* function. For each of the following functions, find whether the function is harmonic, and if so find a conjugate function.

$$\text{(a) } u(x,y) = x^3 + 3x(1+y^2); \qquad \text{(b) } w(x,y) = x^3 + 3x(1-y^2).$$

A3. **(a)** State what it means for a subset $U \subseteq \mathbb{C}$ to be *open*. State what it means for U to be *(path)-connected.*

(b) Sketch each of the following sets, and say (with brief reasons) which of them are open and which are (path)-connected.

$$\text{(a) } \{z \in \mathbb{C} : |z| \geq 1\}; \qquad \text{(b) } \{z \in \mathbb{C} : \Re e(z) \neq \Im m(z)\}.$$

(c) State and prove the *Fundamental Theorem of Path Integrals.*

(d) Let $p : [0, 2\pi] \to \mathbb{C}$ be the path defined by $p(t) = 2 + 2e^{it}$. Sketch this path, and evaluate the following integrals:

$$\text{(a) } \int_p \overline{z}\, dz; \qquad \text{(b) } \int_p \frac{1}{(z-1)^5}\, dz; \qquad \text{(c) } \int_p \frac{e^{z^2}}{z-1}\, dz.$$

A4. **(a)** Find the Taylor series of the function $1/z$ about the point $z = 2$, and state the radius of convergence of the power series.

(b) **(i)** Compute the Laurent series of the function $1/(z^2 - 4)$, about the point $z = 2$, valid in the region $\{z \in \mathbb{C} : 0 < |z - 2| < 4\}$;

(ii) Compute the Laurent series of the same function, about the same point, valid in the region $\{z \in \mathbb{C} : 4 < |z - 2|\}$.

(c) For each of the following functions, state the order of the pole at the point $z = 2$, and find the residue at that point:

$$\text{(a) } \frac{1}{z^2 - 4}; \qquad \text{(b) } \frac{z \cos z}{(z-2)^2}; \qquad \text{(c) } \frac{e^z}{(z^2 - 4)^3}.$$

A5. Use complex variable methods to evaluate:

$$\text{(i) } \int_0^{2\pi} \frac{1 + \cos t}{5 + 4\cos t}\, dt; \qquad \text{(ii) } \int_{-\infty}^{\infty} \frac{1}{x^4 + 1}\, dx.$$

END