MATH-208001

This question paper consists of 3 printed pages, each of which is identified by the reference MATH-208001.

Only approved basic scientific calculators may be used.

Examination for the Module MATH-2080
(January 2008)

Further Linear Algebra

Time allowed : 2 hours

Do not attempt more than **four** questions
All questions carry equal marks

1. **(a)** Let V be a vector space over a field F. What is meant by saying that W is a *vector subspace* of V?

Which of the following are vector subspaces of $\mathbb{R}^3$? Give proofs or counterexamples, as appropriate.

(i) $W_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + 5x_2 + 6x_3 = 0\}$;

(ii) $W_2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 x_2 x_3 = 0\}$.

(b) Let U and W be vector subspaces of V. Show that $U \cap W$ is a vector subspace of V.

Define the subspace $U + W$. What does it mean to say that V is the *direct sum*, $U \oplus W$, of U and W?

(c) Consider the following subspaces of $\mathbb{R}^3$:

$$U = \{(a + 3b, a, b) : a, b \in \mathbb{R}\}, \quad V = \{(c, 0, 0) : c \in \mathbb{R}\}, \quad W = \{(4d, d, d) : d \in \mathbb{R}\}.$$

Giving brief reasons, determine whether or not

(i) $\mathbb{R}^3 = U \oplus V$, (ii) $\mathbb{R}^3 = U \oplus W$, (iii) $\mathbb{R}^3 = V \oplus W$.

2. **(a)** Let V be a vector space over a field F, and let $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$. Define what is meant by saying that the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$

(i) is *linearly independent*; (ii) *spans* V; (iii) is a *basis* of V.

What is meant by the *dimension* of a vector space?

Find a basis for the subspace of $\mathbb{R}^4$ consisting of all the solutions (x, y, z, w) to the pair of equations $x + y - 2w = 0$, $y + z + 3w = 0$.

(b) Which of the following sets are bases for $\mathbb{C}^2$? Give brief explanations.

(i) $\{(1, i), (i, 1)\}$; (ii) $\{(1, i), (i, -1)\}$; (iii) $\{(1, 0), (0, 1), (1, i)\}$.

(c) Show that the set $\{(1, 0, 1, 1),\ (1, 1, 0, 1),\ (0, 1, 1, 0)\}$ is not linearly independent in $\mathbb{F}_2^4$, where $\mathbb{F}_2$ denotes the field of two elements.

Find a basis for the subspace W spanned by this set of vectors. List all the elements of W.

3. **(a)** Let U and V be finite-dimensional vector spaces over a field F. Give the definition of a *linear mapping* $T : U \to V$.

Which of the following formulae define linear mappings from $\mathbb{R}^3$ to $\mathbb{R}^2$? Give a proof or counterexample in each case, as appropriate.

(i) $T_1(x, y, z) = (x^2, y + z)$; (ii) $T_2(x, y, z) = (0, x)$; (iii) $T_3(x, y, z) = (x + 1, y + 1)$.

Question 3 continues . . .

(b) Define the *image (range)* and the *null-space (kernel)* of a linear mapping $T : U \to V$.

Show that the null-space of T is a vector subspace of U.

State the definitions of the *rank*, $r(T)$, and *nullity*, $n(T)$, of T, and give a formula for $r(T) + n(T)$.

Let the linear mapping $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $T(x, y) = (x - y, 2y - 2x, 0)$.

Write down bases for its image and null-space, and determine its rank and nullity.

Find the matrix A that represents T with respect to the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$.

4. **(a)** Let A and B be matrices with entries in the same field F. What is meant by saying that A is *equivalent* to B?

Prove that if A is equivalent to B, then B is equivalent to A.

(b) Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 3 & 6 & 3 \end{pmatrix}$.

Find r and invertible real matrices Q and P such that $Q^{-1}AP = \begin{pmatrix} I_r & O \\ O & O \end{pmatrix}$, where each O denotes a matrix of zeroes (not necessarily the same size in each case).

Paying special attention to the order of the vectors, write down bases of $\mathbb{R}^3$ with respect to which $Q^{-1}AP$ represents the mapping $\mathbf{x} \mapsto A\mathbf{x}$.

(c) What does it mean to say that two vector spaces U and V over the same field are *isomorphic*?

For $k \geq 1$, let P_k denote the vector space of all real polynomials of degree at most k. For which value of n is P_k isomorphic to $\mathbb{R}^n$? Give a brief reason for your answer.

5. **(a)** Let V be a vector space over a field F and let $T : V \to V$ be a linear mapping. Give the definitions of an *eigenvalue* and the corresponding *eigenvector* of T.

Let λ be an eigenvalue of T, and let p be a polynomial with coefficients in F. Define the linear mapping $S = p(T)$ and show that $p(\lambda)$ is an eigenvalue of S.

(b) Let $A = \begin{pmatrix} 7 & -4 & 1 \\ 3 & 0 & 1 \\ -6 & 2 & -1 \end{pmatrix}$.

Given that the characteristic equation of A is $(\lambda - 1)^2(\lambda - 4)$, find its eigenvalues and eigenvectors.

Is A diagonalisable?

What is the *minimum polynomial* of a matrix? Find the minimum polynomial of A.

END