MATH–208001

This question paper consists of 3 printed pages, each of which is identified by the reference MATH–208001

Only approved basic scientific calculators may be used.

Examination for the Module MATH–2080

(January, 2007)

FURTHER LINEAR ALGEBRA

Time allowed : 2 hours

Answer **four** questions. All questions carry equal marks.

1. **(a)** Let V be a vector space over a field F. Give the definition of a *vector subspace* U of V.

Show that $W := \{(a+b, a, b^2) : a, b \in \mathbb{R}\}$ is not a vector subspace of $\mathbb{R}^3$.

(b) Define what is meant by saying that a vector space V over a field F is the *direct sum* of two subspaces U, W.

Let

$$U := \{(a, 2a, 3a) : a \in \mathbb{R}\} \qquad \text{and} \qquad W := \{(x, y, z) : x + y + z = 0, \quad x, y, z \in \mathbb{R}\}.$$

Show that $\mathbb{R}^3 = U \oplus W$.

(c) Let U be a vector subspace of a vector space V over a field F, and suppose that $\boldsymbol{w} \in V$. Show that $U_1 := \{a\boldsymbol{w} + \boldsymbol{u} : a \in F, \boldsymbol{u} \in U\}$ is also a vector subspace of V.

2. **(a)** Let U and V be finite-dimensional vector spaces over a field F. Give the definition of a *linear mapping* $T : U \to V$.

Show that $T : \mathbb{C}^2 \to \mathbb{C}$ given by $T(z_1, z_2) := z_1 + i$ is not a linear mapping.

Show that if $T : U \to V$ and $S : V \to W$ are linear mappings, then the composition $ST : U \to W$ is also a linear mapping.

(b) Define what is meant by the *null-space* and the *range* of a linear mapping $T : U \to V$.

In the case that U is finite-dimensional, state the definitions of *rank* and *nullity* of T. What is then the value of $\operatorname{rank} T + \operatorname{nullity} T$?

(c) Let the linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $T(x, y, z) = (x + y, y + z)$. Write down the matrix A representing T with respect to the standard basis. Find the matrix B which represents T with respect to the (ordered) bases $\{(1, 0, 0), (0, 1, 0), (1, 1, -1)\}$ of $\mathbb{R}^3$ and $\{(1, 0), (0, 1)\}$ of $\mathbb{R}^2$.

3. **(a)** Let A and B be matrices with entries in a field F. What is meant by saying that A is *equivalent* to B?

Prove that if A is equivalent to B and if B is equivalent to C, then A is equivalent to C.

(b) Let

$$A := \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix},$$

where the entries of A are in the 2 element field $\mathbb{F}_2$.

Find r and invertible matrices Q and P (with entries in $\mathbb{F}_2$) such that

$$QAP = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

Find Q^{-1} and write down bases of $\mathbb{F}_2^4$ and $\mathbb{F}_2^5$ with respect to which QAP represents the mapping $\boldsymbol{x} \mapsto A\boldsymbol{x}$.

4. **(a)** Let A and B be $n \times n$ matrices with entries in a field F. What is meant by saying that A is *similar* to B?

Show that if A is similar to B, then A^2 is similar to B^2.

(b) Define an *eigenvalue* of an $n \times n$ matrix A of real numbers. Give also the definitions of the corresponding *eigenvectors* and *generalised eigenvectors*.

The matrix

$$A := \begin{pmatrix} 2 & 1 & 1 \\ -1 & 3 & 0 \\ 1 & -1 & 2 \end{pmatrix}.$$

has characteristic equation $(\lambda - 3)(\lambda - 2)^2 = 0$

Find the eigenvectors and generalised eigenvectors of A.

Hence, or otherwise, write down a Jordan canonical matrix B which is similar to A and find a non-singular matrix P such that $B = P^{-1}AP$.

5. **(a)** Write down the minimum polynomial of the matrix

$$A := \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Question 5 continues . . .

(b) Let $(V, \langle\,,\rangle)$ be a real inner-product space.

Define the *norm* $\|\boldsymbol{v}\|$ of a vector in V.

State the Cauchy Schwarz inequality for vectors $\boldsymbol{u}, \boldsymbol{v} \in V$.

Show that, for any vectors $\boldsymbol{u}, \boldsymbol{v} \in V$ we have

$$\|\boldsymbol{u} + \boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|.$$

(c) Define what it means for a vector $\boldsymbol{u}$ to be *orthogonal* to a vector $\boldsymbol{v}$ in an inner-product space.

Show that if $\boldsymbol{u}$ is orthogonal to $\boldsymbol{v}$, then

$$\|\boldsymbol{u} + \boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2.$$

(d) Let $\mathbb{R}^4$ have the standard inner-product. Given that $\{(1,-1,1,-1), (1,-1,0,0), (0,1,-1,0)\}$ is a basis of $E = \{(x,y,z,t) : x+y+z+t = 0\}$, find an orthonormal basis of E.

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