MATH–208001

This question paper consists of 3 printed pages, each of which is identified by the reference MATH–208001

Only approved basic scientific calculators may be used.

Examination for the Module MATH–2080
(January, 2006)

FURTHER LINEAR ALGEBRA

Time allowed : 2 hours

Answer **four** questions. All questions carry equal marks.

1. **(a)** Let V be a vector space over a field F. Let $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n \in V$. Define what is meant by saying that the set $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$

(i) is *linearly independent*;
(ii) *spans* V;
(iii) is a *basis* of V.

Let $V = \{\boldsymbol{x} \in \mathbb{R}^4 : x_1 + x_2 - x_3 - x_4 = 0\}$. Show that $\{(1, -1, 0, 0), (1, 0, 1, 0), (1, 0, 0, 1)\}$ is a basis of V.

(b) Let W_1, W_2 be vector subspaces of a vector space U over a field F. Define what is meant by saying that U is the *direct sum*, $W_1 \oplus W_2$, of W_1 and W_2.

Prove that, if $U = W_1 \oplus W_2$, then each $\boldsymbol{u} \in U$ has a unique decomposition of the form $\boldsymbol{u} = \boldsymbol{w}_1 + \boldsymbol{w}_2$.

Let V be the subspace of $\mathbb{R}^4$ defined in **(a)**. Find a vector subspace W of $\mathbb{R}^4$ such that $\mathbb{R}^4 = V \oplus W$.

2. **(a)** Let U and V be finite-dimensional vector spaces over a field F and let $T : U \to V$ be a linear mapping. Define what is meant by the *null-space*, the *range*, the *nullity*, and the *rank* of T.

Prove that the null space, $\ker T$, is a vector subspace of U.

What is the value of $\operatorname{rank} T + \operatorname{nullity} T$?

(b) Which of the following are linear mappings? Justify your answers. For each one that is a linear mapping, find a basis for its null space and hence find its nullity and rank.

(i) $S : \mathbb{R}^2 \to \mathbb{R}, \qquad S(x_1, x_2) = x_1 x_2\,;$
(ii) $T : \mathbb{C}^3 \to \mathbb{C}^2, \qquad T(z_1, z_2, z_3) = (z_1 + i z_2, z_3)\,.$

3. **(a)** Let A and B be matrices with entries in a field F. What is meant by saying that A is *equivalent* to B?

Prove that if A is equivalent to B, then B is equivalent to A.

(b) Let

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \end{pmatrix},$$

where the entries of A are in the 2 element field $\mathbb{F}_2$.

Find r and invertible matrices Q and P (with entries in $\mathbb{F}_2$) such that

$$QAP = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

Find Q^{-1} and write down bases of $\mathbb{F}_2^4$ and $\mathbb{F}_2^5$ with respect to which QAP represents the mapping $\boldsymbol{x} \mapsto A\boldsymbol{x}$.

4. **(a)** Let V be a vector space over a field F and let $T : V \to V$ be a linear mapping. State the definitions of an *eigenvalue* and *corresponding eigenvector* of T.

Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ be eigenvectors corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ of a linear mapping $T : V \to V$. Show that the set $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ is linearly independent.

(b) Show that 2 is the only eigenvalue of the matrix

$$A = \begin{pmatrix} 3 & 0 & -1 \\ 3 & 1 & -2 \\ -1 & 1 & 2 \end{pmatrix}.$$

Find the eigenvectors and generalised eigenvectors of A

Hence, or otherwise, write down the Jordan canonical matrix B which is similar to A.

5. **(a)** Find the minimum polynomial of the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Question 5 continues . . .

(b) Let V be a vector space over $\mathbb{R}$. Define what is meant by an *inner product* on V.

Show that the following formula does not define an inner-product on $\mathbb{R}^2$:

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = x_1 y_1 + x_2 y_2 + 1$$

(where $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{y} = (y_1, y_2)$).

Define what is meant by saying that a set $\{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_n\}$ is an *orthonormal basis* of V.

Let $\mathbb{R}^3$ have the standard inner product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3$. Find an orthonormal basis of the subspace V given by $V = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 - x_2 + x_3 = 0\}$.

END