MATH–208001

This question paper consists of 3 printed pages, each of which is identified by the reference MATH–208001

Only approved basic scientific calculators may be used.

Examination for the Module MATH–2080

(January, 2005)

FURTHER LINEAR ALGEBRA

Time allowed : 2 hours

Answer **four** questions. All questions carry equal marks.

1. Let V be a vector space over a field F. Define what is meant by saying that a set W is a *vector subspace* of V.
Let U, W be vector subspaces of V. Show that

$$U + W := \{\boldsymbol{u} + \boldsymbol{w} : \boldsymbol{u} \in U \text{ and } \boldsymbol{w} \in W\}$$

is a vector subspace of V.
Define what is meant by saying V *is the direct sum of* U *and* W.

Let W_1, W_2, W_3, and W_4 be the following subspaces of $\mathbb{R}^4$:

$$W_1 = \{(a, b, 2a, b - a) : a, b \in \mathbb{R}\},$$
$$W_2 = \{(c, d, 3c, 2d - c) : c, d \in \mathbb{R}\},$$
$$W_3 = \{(x, y, z, w) \in \mathbb{R}^4 : x = z \text{ and } y = w\},$$
$$W_4 = \{(x, y, z, w) \in \mathbb{R}^4 : x = y \text{ and } z = w\}.$$

Determine whether or not (i) $\mathbb{R}^4 = W_1 \oplus W_2$, (ii) $\mathbb{R}^4 = W_3 \oplus W_4$.

2. **(a)** Let V and W be finite-dimensional vector spaces over a field F and let $T : V \to W$ be a linear mapping. Define what is meant by the *null-space*, the *range*, the *nullity*, and the *rank* of T.

Let $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_r, \boldsymbol{e}_{r+1}, \ldots, \boldsymbol{e}_n\}$ be a basis of V such that $\{\boldsymbol{e}_{r+1}, \ldots, \boldsymbol{e}_n\}$ is a basis of $\ker T$. Show that $\{T(\boldsymbol{e}_1), \ldots, T(\boldsymbol{e}_r)\}$ is a basis of $\operatorname{range} T$ and deduce that

$$\operatorname{rank} T + \operatorname{nullity} T = \dim V.$$

(b) Which of the following are linear mappings? Justify your answers. For each one that is a linear mapping, find a basis for its null space and hence find its nullity and rank.

(i) $T : \mathbb{R}^2 \to \mathbb{R}$, $\quad T(x_1, x_2) = x_1 - x_2$;

(ii) $T : \mathbb{C}^3 \to \mathbb{C}^2$, $\quad T(z_1, z_2, z_3) = (z_1 + iz_2, 2 + z_3)$.

continued ...

3. **(a)** Let V and W be distinct vector spaces over a field F and let $T : V \to W$ be a linear mapping. Suppose that $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ is a basis of V and $\{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_k\}$ is a basis of W. Explain what is meant by the *matrix A of T with respect to the given bases.*

Using a result stated in question 2, or otherwise, show that there are bases of the spaces V and W such that the matrix A representing T takes the form

$$A = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},$$

for an appropriate value of r.

(b) Let

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix},$$

where the entries of A are in the 2 element field $\mathbb{F}_2$.

Find r and invertible matrices Q and P (with entries in $\mathbb{F}_2$) such that

$$QAP = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

[You need not show that P and Q are invertible.]

4. **(a)** Explain what is meant by saying that $n \times n$ matrices A and B are *similar.*

Prove that, if A and B are similar matrices with real entries, then they have the same characteristic polynomials.

(b) Find the eigenvectors and generalised eigenvectors of the mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$, defined by $T(\boldsymbol{x}) = A\boldsymbol{x}$, where

$$A = \begin{pmatrix} 3 & -2 & -1 \\ 0 & 1 & 1 \\ 4 & -4 & -3 \end{pmatrix},$$

given that the characteristic equation of A is $(\lambda - 1)^2(\lambda + 1) = 0$.

Hence, write down a matrix B in Jordan normal form which is similar to A, and find a matrix P such that $B = P^{-1}AP$.

5. **(a)** Let V be a vector space over $\mathbb{R}$. Define what is meant by an *inner product on V.*

(b) Let V be an inner-product space over $\mathbb{R}$. Define what is meant by saying that a set $\{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_n\}$ is an *orthonormal basis* of V.

Question 5 continues . . .

Show that, for such a basis,

$$\boldsymbol{v} = \sum_{i=1}^{n} \langle \boldsymbol{v}, \boldsymbol{f}_i \rangle \boldsymbol{f}_i, \qquad \text{for all } \boldsymbol{v} \in V.$$

(c) Let the mapping $T : \mathbb{R}^4 \to \mathbb{R}^4$ be defined by $T(\boldsymbol{x}) = A\boldsymbol{x}$, where

$$A = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}.$$

Find an orthonormal basis of $\mathbb{R}^4$ (with the standard inner-product) such that T is represented by a diagonal matrix with respect to that basis.

Write down the diagonal matrix.

END