MATH–208001

This question paper consists of 2 printed pages, each of which is identified by the reference MATH–208001

Only approved basic scientific calculators may be used.

Examination for the Module MATH–2080
(January, 2003)

FURTHER LINEAR ALGEBRA

Time allowed : 2 hours

Answer **four** questions. All questions carry equal marks.

1. (a) Let W_1, W_2 be subspaces of the vector space V. Prove that the following conditions are equivalent:

(i) $V = W_1 + W_2$ and $W_1 \cap W_2 = \{\mathbf{0}\}$;

(ii) every vector $\mathbf{v}$ in V can be expressed uniquely as a sum $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$, with $\mathbf{w}_1 \in W_1$ and $\mathbf{w}_2 \in W_2$.

(b) Let W_1, W_2, W_3 and W_4 be the following subspaces of $\mathbb{R}^4$:

$$W_1 = \{\lambda(2,0,1,2) + \mu(0,1,-2,0) : \lambda, \mu \in \mathbb{R}\},$$
$$W_2 = \{(a,b,c,d) \in \mathbb{R}^4 : a + 2c = 0 \text{ and } 2b - d = 0\},$$
$$W_3 = \{(a,b,c,d) \in \mathbb{R}^4 : 3b - d = 0\},$$
$$W_4 = \{\nu(0,2,0,1) : \nu \in \mathbb{R}\}.$$

Determine whether or not (i) $\mathbb{R}^4 = W_1 \oplus W_2$, (ii) $\mathbb{R}^4 = W_1 \oplus W_3$, (iii) $\mathbb{R}^4 = W_3 \oplus W_4$.

2. (a) Let $T : V \to W$ be a linear transformation of vector spaces. Define what is meant by the *kernel of* T and the *image of* T. Prove that the kernel of T is a subspace of V, and that the image of T is a subspace of W.

(b) Which of the following are linear transformations? Justify your answers. For each one that is a linear transformation, find a basis for its kernel and a basis for its image, and hence find the dimensions of those subspaces.

(i) $T : \mathbb{R}^3 \to \mathbb{R}^4$, $\quad T(a,b,c) = (a,0,b,0)$;

(ii) $T : \mathbb{R}^2 \to \mathbb{R}^3$, $\quad T(a,b) = (a-b,0,3b)$;

(iii) $T : \mathbb{R}^3 \to \mathbb{R}^2$, $\quad T(a,b,c) = (1,1)$.

3. (a) Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be a basis for a vector space V, and let $T : V \to V$ be a linear transformation. Explain what is meant by the *matrix* A *of* T *with respect to the given basis.*

Question 3 continues ...

(b) Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(a, b, c) = (c, b, a)$. Write down the matrix A of T with respect to the standard basis. Find the matrix P such that $B = P^{-1}AP$ represents T with respect to the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ where $\mathbf{u}_1 = (1, 0, 1)$, $\mathbf{u}_2 = (0, 1, 1)$, $\mathbf{u}_3 = (0, 1, 0)$. Hence find B.

(c) Find the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} 4 & -2 \\ 2 & 0 \end{pmatrix}.$$

4. **(a)** Explain what is meant by the *characteristic polynomial* and the *minimum polynomial* of a square matrix A. Prove that, if A and B are similar matrices, then they have the same characteristic and minimum polynomials.

(b) You are given that the matrix

$$A = \begin{pmatrix} 7 & 1 & 1 & 1 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 6 & -2 \\ -1 & 1 & -1 & 7 \end{pmatrix}$$

has characteristic polynomial $\phi_A(x) = (x-6)^2(x-8)^2$. Find its Jordan canonical form and its minimum polynomial.

5. **(a)** Let V be a vector space. Define what is meant by an *inner product on* V.

(b) Let $(V, \langle\, ,\, \rangle)$ be an inner product space. Define what is meant by the *length* $\|\mathbf{v}\|$ *of a vector* $\mathbf{v} \in V$, and show that it satisfies the following properties for $\mathbf{u}$, $\mathbf{v} \in V$ and $\alpha \in \mathbb{R}$:

(i) $\|\alpha\mathbf{v}\| = |\alpha|\|\mathbf{v}\|$;

(ii) $\|\mathbf{v}\| > 0$ if $\mathbf{v} \neq \mathbf{0}$ and $\|\mathbf{v}\| = 0$ if $\mathbf{v} = \mathbf{0}$;

(iii) $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.

[You may quote the Schwarz inequality for the inner product without proof.]

(c) Let $(V, \langle\, ,\, \rangle)$ be an inner product space. Explain what is meant by (i) an *orthogonal basis*, (ii) an *orthonormal basis* of V.

Let V the space $\mathbb{R}^4$ with its usual inner product. Use the Gram-Schmidt process to find an orthonormal basis for the subspace W spanned by the three vectors $(1, 2, 0, 1)$, $(0, 1, 0, -1)$, $(0, -1, 2, 0)$.

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