This question paper consists of 4 printed pages,
each of which is identified by the reference MATH-2051.
Only approved basic scientific calculators may be used.

Examination for the module MATH-2051
(January 2007)

GEOMETRY OF CURVES AND SURFACES

Time allowed : 2 hours

Answer **four** questions. All questions carry equal marks.

1. (a) Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be given by $\gamma(t) = (t^3, \sin t, t + 2t^2)$. Show that γ is a regular parametrized curve.

   (b) Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be given by $\gamma(t) = (1+3\cos t, 2t-1, 3\sin t)$. Calculate the arc-length along γ from $t = -1$ to $t = 2$.

   (c) Let $\gamma : (0, \infty) \to \mathbb{R}^2$ be given by $\gamma(t) = (t^2, t^5 - t^3)$.

   (i) Calculate the speed of γ.

   (ii) Calculate the unit tangent vector $T(t)$ and unit normal vector $N(t)$.

   (iii) What is the angle between T and N?

   (iv) Compute $\gamma''(t)$ and hence the signed curvature of γ.

   (v) Determine the proper inflexion points of γ.

2. (a) Let $\alpha(t) = (3 + 2\sin(t), 1 - 2\cos(t), 5 - t)$. Construct the Frenet frame $[T(\pi/2), N(\pi/2), B(\pi/2)]$ for α at time $t = \pi/2$.

   (b) Let $\gamma : I \to \mathbb{R}^3$ be a unit speed space curve of nonvanishing curvature. Suppose γ has Frenet frame $[T(t), N(t), B(t)]$.

   (i) Write down the Serre-Frenet formula for $[T'(t), N'(t), B'(t)]$, defining any functions you use.

   (ii) Prove that the formula for $N'(t)$ is correct (assuming those for T' and B' are correct).

Q2 continues . . .

(c) Let α be the space curve in part (a). Find $N'(t)$ and hence given that curvature at $t = \pi/2$ is $2/\sqrt{5}$ find the torsion for α at $t = \pi/2$. Is there enough information to decide whether the curve is planar or not? Explain your answer.

3. (a) Let $\gamma : (a, b) \to \mathbb{R}^2$ be a smooth parametrized curve. Define the *involute* of γ at $t_0 \in (a, b)$, which we shall denote by I_γ.

(b) Suppose that γ above is unit speed and we denote the signed curvature of γ by $\kappa_\gamma^\pm(t)$.

(i) Calculate $|I_\gamma'(t)|$, $T_{I_\gamma}(t)$ and $N_{I_\gamma}(t)$ in terms of functions and vectors associated to γ. You may assume the formulas $T_\gamma'(t) = \kappa_\gamma^\pm(t) N_\gamma(t)$ and $N_\gamma'(t) = -\kappa_\gamma^\pm(t) T_\gamma(t)$ for a unit speed curve γ.

(ii) Hence show that, for $t \neq t_0$, the signed curvature $\kappa_I^\pm$ of I_γ is given by

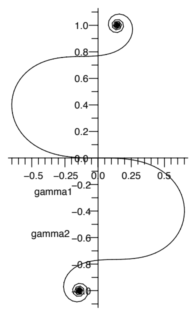
$$\kappa_I(t) = \frac{\operatorname{sign}\left(\kappa_\gamma^\pm(t)\right)}{|t - t_0|}.$$

(c) Given a prescribed smooth function $\kappa : \mathbb{R} \to \mathbb{R}$ there exists a unique unit speed curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ with $\gamma(0) = (0, 0)$, $\gamma'(0) = (1, 0)$ and signed curvature κ. The curves corresponding to the following signed curvature functions

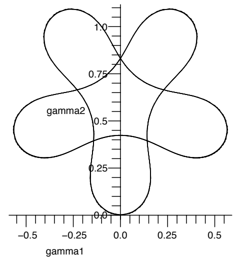
(a) t^2 (b) $2 + 5\cos(5t)$ (c) $-(4t^4 - 3t^2 + 2)$
(d) $4t^3 - 3t^2 + 2$ (e) $2t^3 - 5t$

are depicted on the next page in the wrong order. Determine which curve corresponds to which signed curvature. Justify your answers.

Q3 continues ...



(i)



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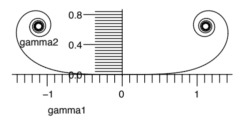

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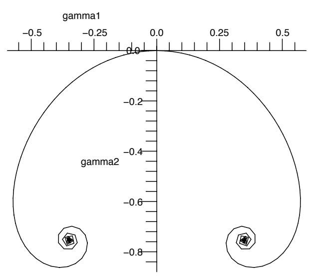

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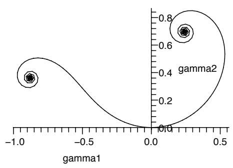

(v)

4. Let $M : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $M(x,y) = (y, x+y^2, y+\cos(xy))$.

   (a) Define the term *regular parametrized surface* and show that M above is a regular parametrized surface.

   (b) Let $p = (2,4,3) = M(0,2)$. Determine whether the following vectors are tangent to M, normal to M or neither.

   (i) $u = (1,4,-2)$, (ii) $u = (3,1,3)$.

   (c) Let $f(y_1, y_2, y_3) = y_1^3 - 2y_2$. Given that $w = (2,5,2)$ is a tangent vector at $p = (2,4,3)$, compute the directional derivative $\nabla_w f$.

Continued ...

5. Suppose that M is a regular parametrized surface and $p \in M$.

   (a) Define

      (i) the shape operator at p,

      (ii) the normal curvature at p,

      (iii) the principal curvatures at p, and

      (iv) the principal curvature directions at p.

   (b) Suppose that $p \in M$. The matrix representing the shape operator at p of a surface M with respect to the coordinate basis $[\epsilon_1, \epsilon_2]$ is

$$\widehat{S}_p = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}.$$

   Determine the principal curvatures κ_1, κ_2 and the corresponding *unit* vectors that give the principal directions u_1 and u_2.

   Determine the Gauss and Mean curvature at p for this M.

   (c) Let $k_p : U_pM \to \mathbb{R}$ denote the normal curvature function at p. In each of the following cases *either* construct a tangent vector $v \in T_pM$ with the specified properties *or* explain why no such vector exists.

      (i) A unit vector v with $k_p(v) = 5$.

      (ii) A unit vector v with $k_p(v) = -1$.

      (iii) A vector v with $S_p(v) = 2v$.

      (iv) A vector v with $k_p(v) = 3$.

The End