MATH–202101

Only approved basic scientific calculators may be used.

This question paper consists of 2 printed pages, each of which is identified by the reference MATH–2021

Examination for the Module MATH–2021

(May–June 2001)

COMPLEX ANALYSIS

Time allowed : 2 hours

Do not answer more than four questions. All questions carry equal marks.

1. State what is meant by the *radius of convergence* of a complex power series $\sum a_n z^n$, and calculate the radii of convergence of the following series:

$$\textbf{(a)} \quad \sum_{n=0}^{\infty} z^{2^n}; \qquad \textbf{(b)} \quad \sum_{n=0}^{\infty} n! z^n; \qquad \textbf{(c)} \quad \sum_{n=0}^{\infty} 2^n z^n; \qquad \textbf{(d)} \quad \sum_{n=1}^{\infty} \frac{z^{2n}}{(2n-1)(n!)}.$$

A power series $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $R > 0$ and defines a function $f(z)$ on $\{z \in \mathbb{C} : |z| < R\}$. Write down the power series for the functions $f'(z)$ and $f(2z)$, and state without proof their radii of convergence.

Show that there is an entire function $f : \mathbb{C} \to \mathbb{C}$, expressible as the sum of a power series, such that

$$f(0) = 0, \quad f'(0) = 0, \quad \text{and} \quad f''(z) = \exp(z^2) \quad \text{for all} \quad z \in \mathbb{C}.$$

2. **(a)** Let U be an open set in $\mathbb{C}$ and let $f : U \to \mathbb{C}$ be an analytic function. Suppose that $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$, and u and v are real-valued functions. Prove that u and v satisfy the Cauchy–Riemann equations.

(b) Given also that u and v are C^2 functions, deduce that they are harmonic.

(c) For each of the following functions determine whether it is the real part of an analytic function defined on $\mathbb{C}$. If it is, determine a corresponding imaginary part.

(i) $u(x, y) = x^2 - y^2 + 2xy$;

(ii) $u(x, y) = x^3 - y^3 + 3x^2 y$.

3. (a) State Liouville's Theorem on entire functions and the Maximum Modulus Principle.

Let $p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$ be a nonconstant polynomial. Show that there is a number $R > 0$ such that

$$|p(z)| \geq \frac{|z|^n}{2} \qquad \text{for} \quad |z| \geq R.$$

By considering $f(z) = \dfrac{1}{p(z)}$ and using the results above, deduce that p has a zero in $\mathbb{C}$.

(b) Write down entire functions $g(z)$, $h(z)$ such that:

(i) $g(z)$ has no zeroes in $\mathbb{C}$;

(ii) $h(z)$ has infinitely many zeroes in $\mathbb{C}$.

Briefly justify your answers.

4. (a) Explain what a *contour* in $\mathbb{C}$ is, and define its *length.*

Give the definition of $\displaystyle\int_\gamma f(z)\,dz$ for a continuous function f on a contour γ.

(b) Evaluate the following integrals, where in each case γ is the unit circle taken once anticlockwise. (Standard results may be quoted without proof.)

(i) $\displaystyle\int_\gamma \frac{\cos(1/z)}{z^2}\,dz;$

(ii) $\displaystyle\int_\gamma |z+2|^2\,dz;$

(iii) $\displaystyle\int_\gamma \tan z\,dz;$

(iv) $\displaystyle\int_\gamma \frac{\exp(z^2)}{z(z+3)}\,dz.$

5. Calculate the following integrals using complex variable techniques.

(a) $\displaystyle\int_0^{2\pi} \frac{d\theta}{1 + 8\cos^2\theta};$

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x\,dx}{2x^4 + 5x^2 + 2}.$

END