MATH171501

This question paper consists of 4 printed pages, each of which is identified by the reference **MATH1715**.

Statistical tables are provided. Only approved basic scientific calculators may be used.

Examination for the Module MATH1715
(January 2003)

INTRODUCTION TO PROBABILITY

Time allowed: **2 hours**

Do not attempt more than FOUR questions.

All questions carry equal marks.

1. (a) Consider a class consisting of 30 Sixth Form students. Suppose that both French and Maths are studied by 6 students, both German and Maths are studied by 9 students, while the proportion of students doing Maths and both foreign languages equals 10%. If a student is picked up from the class at random, what is the probability that he is doing Maths and at least one foreign language? Draw an appropriate Venn diagram, using letters F, G and M for French, German and Maths, respectively.

(b) (i) Give the mathematical definition of *mutual independence* of three events, A, B and C.

(ii) Suppose that two fair coins are tossed once. Consider the events

$$\begin{aligned} A &= \{\text{The first coin shows heads}\}, \\ B &= \{\text{The second coin shows heads}\}, \\ C &= \{\text{Both coins show the same}\}. \end{aligned}$$

Verify that these events are pairwise independent. Are they mutually independent?

(c) An urn contains five red balls and five black balls. Players A and B draw balls from the urn alternately, without replacement, until someone draws a red ball and wins the game. If player A draws first, what is the probability that he will win the game?

(d) An elevator in a building starts with 4 people and stops at 7 floors. If each passenger is equally likely to get off at any floor, independently of the others, what is the probability that at least two passengers get off at the same floor? Give the answer to 3 significant figures.

2. (a) Two boxes contain 3 white dice and 2 black dice each. A die is chosen at random from the first box and put into the second box. What is now the probability of drawing a white die from the second box?

(b) A blood test for hepatitis is 90% effective in detecting the disease when it is in fact present. However, the test yields a "positive" result for 1% of healthy persons tested. The disease rate in the general population is 1 in $10,000$.

(i) What is the probability that a person who receives a positive test result actually has hepatitis?

(ii) A patient is sent for a blood test because he has lost his appetite and has developed jaundice (that is, yellowness of the skin and the whites of the eyes). The physician knows that this type of patient has a probability of $1/2$ of having hepatitis. If this patient gets a positive result on his blood test, what is the probability that he has hepatitis?

Give your answers to 3 significant figures.

QUESTION 2 CONTINUED...

(c) Let S_n denote the number of successes in n Bernoulli trials with probability p of success. Explain how the random variable S_n can be represented as $S_n = X_1 + \cdots + X_n$, and use this representation to derive the expected value $\mathsf{E}(S_n) = np$. State clearly the properties of expectation that you use.

(d) A football team loses each game with probability 0.2, independently of the outcomes of other games.

(i) What is the probability that the team will lose at most 2 games out of ten? (Give the answer to 3 significant figures.)

(ii) What is the expected number of wins in ten games?

(iii) How many games would you expect the team to win before they first lose?

3. (a) A group of 6 men and 6 women is randomly divided into two groups of size 6 each. What is the probability that both groups will have the same numbers of men?

(b) Suppose that 1% of a certain brand of Christmas light bulbs are defective. Use the Poisson approximation to compute the probability that in a box of 25 there will be at most one defective bulb. (Give the answer to 3 significant figures.)

(c) The train runs every hour and stays at the station for 6 minutes. You have forgotten the timetable, so you arrive at the station at random, so that your arrival time is uniformly distributed over the one-hour gap between two consecutive trains.

(i) Show that the probability of seeing the train when you arrive at the station equals 0.1.

(ii) Find the probability that you will have to wait for the train no longer than 10 minutes.

(d) Suppose that an adult's weight has a normal distribution $\mathrm{N}(\mu, \sigma^2)$ with parameters $\mu = 68.5\,\mathrm{kg}$ and $\sigma = 5.0\,\mathrm{kg}$. Using the tables of the standard normal distribution, find the probability (to 3 significant figures) that a randomly picked person will weigh between 60.3 and 77.2 kg. [6]

4. (a) For two discrete random variables X and Y with joint probability mass function $p_{XY}(x, y)$, what is meant by the *marginal distributions* of X and Y? Write down the formulae to compute the marginal distributions $p_X(x)$ and $p_Y(y)$. [2]

(b) Two fair dice are rolled once. Let X be the value obtained on the first die and Y be the smaller of the two values on the dice.

(i) Find the joint probability mass function of X and Y. [5]

(ii) Verify that the marginal probability mass function of X is given by [2]

$$p_X(x) = \frac{1}{6}, \qquad x = 1, 2, \ldots, 6.$$

(iii) Determine the marginal distribution of Y. [2]

(iv) Find the expected value of $X - Y$. [3]

(c) Let U and V be independent random variables such that $\mathsf{E}(U) = 0$, $\mathsf{E}(V) = 1$, $\mathsf{Var}(U) = 2$, and $\mathsf{Var}(V) = 1$. Consider the new random variable, $W = U - 2V$. Find the variance of W and the covariance $\mathsf{Cov}(U, W)$. [6]

5. (a) A function $f(x)$ is defined by the formula $f(x) = Cx^2$ for $x \in [0, 2]$ and $f(x) = 0$ elsewhere. Find the constant C so that $f(x)$ be a probability density function. [2]

(b) Suppose that a random variable X is exponentially distributed with probability density function given by

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if} \quad x \geq 0, \\ 0 & \text{if} \quad x < 0. \end{cases}$$

(i) Find the corresponding cumulative distribution function $F_X(x)$ and sketch its graph. [3]

(ii) If the random variable Y is defined as $Y = \sqrt{X}$, obtain its cumulative distribution function, $F_Y(y)$, and probability density function, $f_Y(y)$, and sketch their graphs. [7]

(c) Suppose that the waiting time in a queue has an exponential distribution with parameter λ, such that $1/\lambda = 20$ minutes.

(i) What is the probability that standing in the queue will take more that 20 minutes? [3]

(ii) Given that 20 minutes have already been spent in the queue, what is the probability that the overall waiting time will be at least one hour? [5]

Give your answers to 3 significant figures.

END