MATH171501

This question paper consists of 4 printed pages, each of which is identified by the reference **MATH171501**.

New Cambridge Elementary Statistical Tables are provided. Only approved basic scientific calculators may be used.

Examination for the Module MATH1715
(January 2004)

INTRODUCTION TO PROBABILITY

Time allowed: **2 hours**

Attempt ALL questions in Section A and THREE questions from Section B.

For Section A only write down a single letter answer for each question.
Section A is worth 40% of the examinations marks.

All questions within each section carry equal marks.

Section A

A1. Suppose that 60% of adults over 30 in a certain community are graduates. Furthermore, suppose that 80% of the graduates over 30 have incomes over $\pounds 15,000$. What percentage of adults over 30 in this community are graduates and have incomes over $\pounds 15,000$?

A: 48   B: 20   C: 40   D: 12   E: 32

A2. Suppose that the sex of a child can be modelled as independent rolls of a 35-sided die, with 18 faces marked 'boy' and the other 17 marked 'girl'. Under a stopping rule: "stop after the first son", what proportion of completed families would comprise at least four children?

A: 0.115   B: 0.056   C: 0.070   D: 0.486   E: 0.136

A3. Suppose that the probability of being born on Christmas day is $1/365$. If we have a group of 100 people (whose birthdays are independent) then what is the probability that no more than one person in the group has a Christmas birthday:

A: 0.209   B: 0.969   C: 0.791   D: 0.031   E: 0.726

A4. Assume that major earthquakes occur world-wide at random but at an estimated average rate of once every 437 days. Assuming a continuous-time model for the phenomenon find the standard deviation of the time in days between successive earthquakes.

A: 20.9   B: 218   C: 437   D: 190969   E: 0.0023

A5. Blood plasma levels in smokers (measured in *ng/ml.*) can be modelled as $T \sim N(315, 17161)$. What proportion of smokers has nicotine levels between 300 and 500?

A: 1.3754   B: 0.9210   C: 0.4544   D: 0.4666   E: 0.0046

A6. On any given day of the year (365 days), past experience suggests that the probability of rain is about 0.3. Assuming that each day is independent, what is the approximate probability that in a whole year it rains at least 100 times?

A: 0.86   B: 0.73   C: 0.91   D: 0.87   E: 0.30

A7. The number of cars passing through a certain toll gate at a certain bridge has a Poisson distribution, with the number of cars averaging 0.3 per minute. Find the probability of more than one car passing through this toll gate during a given 5-minute period.

A: 0.037   B: 0.259   C: 0.442   D: 1.50   E: 0.777

A8. If the continuous random variable X has probability density function given by

$$f(x) = \begin{cases} 1 & \text{for } 0 < x \leq 0.5 \text{ and } 1 < x \leq 1.5 \\ 0 & \text{otherwise} \end{cases}$$

what is the variance of X?

A: 0.520 B: 0.750 C: 0.866 D: 0.271 E: 0.833

A9. An urn contains three chips, two marked Failure and one marked Success. Players A and B take turns drawing a single chip from the urn, each chip being returned to the urn before the next player draws. The winner of the game is the first person to draw Success. The game continues until there is a winner. If A draws first, what is the probability of A winning the game?

A: $1/2$ B: $3/5$ C: $1/3$ D: $2/3$ E: $5/9$

A10. A man aiming at a target receives 10 points if his shot is within 1cm of the centre of the target, five points if it is between 1cm and 3cm, and three points if it is between 3cm and 5cm. What is the expected number of points scored if the man's shot is uniformly distributed in a circle of radius 8cm centred on the target?

A: 36/8 B: 26/8 C: 36 D: 18 E: 49/32

Section B

B1. (a) At a dance, n men and their n wives are paired at random. Derive the expected number of men dancing with their wives, stating any results that you use.

(b) Let X be a geometrically distributed random variable, *i.e.* $X \sim$ Geometric(p), and let $M > 0$ be an integer. If $Z = \max(X, M)$, find the mean of Z. Check that $\mathsf{E}(Z) = 1/p$ when $M = 1$.

B2. Suppose an experiment has r possible outcomes $1, 2, \ldots, r$. Suppose also that outcome s occurs with probability $p_s, s = 1, \ldots, r$. The experiment is repeated n times.

Let X be the number of times that the first outcome occurs, and let Y be the number of times that the the second outcome occurs.

(i) Let $I_i = 1$ if the ith trial yields outcome 1, and let $I_i = 0$ otherwise. Similarly, let $J_i = 1$ if the ith trial yields outcome 2, and let $J_i = 0$ otherwise. Show that $\mathsf{E}(I_i J_i) = 0$ and that $\mathsf{E}(I_i J_j) = p_1 p_2$ when $i \neq j$.

(ii) Noting that $X = \sum_i I_i$ and $Y = \sum_i J_i$, show that $\mathsf{E}(XY) = n(n-1)p_1 p_2$ and obtain $\mathsf{E}(X)$ and $\mathsf{E}(Y)$.

(iii) By first calculating $\mathsf{Cov}(X, Y), \mathsf{Var}(X)$ and $\mathsf{Var}(Y)$, show that

$$\mathsf{Corr}(X, Y) = \rho(X, Y) = -\sqrt{\frac{p_1 p_2}{(1-p_1)(1-p_2)}}.$$

B3. (a) Suppose that the weight of a person selected at random from some population is normally distributed with parameters μ and σ. Suppose also that $\mathsf{P}(X \leq 80\text{Kg}) = 1/2$ and $\mathsf{P}(X \geq 70\text{Kg}) = 3/4$. Find μ and σ and find $\mathsf{P}(X \geq 100\text{Kg})$. Of all the people in the population weighing at least 100Kg, what percentage will weigh over 110Kg?

(b) Suppose a very large number of identical radioactive particles have decay times which are exponentially distributed with some parameter λ. If one half of the particles decay during the first second, how long will it take for 75% of the particles to decay?

B4. (a) If X is a Binomial random variable with parameters n and p, give an expression for the probability distribution $\mathsf{P}(X = r)$ and state the sample space of X.

Obtain the value of r which maximizes the probability $\mathsf{P}(X = r)$ (in terms of n and p).

Using the fact that the probability generating function of X is $P_X(s) = ((1-p)+ps)^n$, or otherwise, **show** that the mean and the variance of X are given by np and $np(1-p)$, respectively.

(b) The continuous random variable X has probability density function given by

$$f(x) = \begin{cases} x/c & 0 \leq x < c \\ 1 & c \leq x < 2c \\ -x/c + 3 & 2c \leq x < 3c \\ 0 & \text{otherwise.} \end{cases}$$

Sketch this density function, evaluate the constant c, and calculate the mean of X.

END