MATH–133101

This question paper consists of 3 printed pages, each of which is identified by the reference MATH–133101

Only approved basic scientific calculators may be used.

Examination for the Module MATH–1331

(May/June 2003)

Linear algebra with applications

Time allowed : 3 hours

Answer **four** questions.
All questions carry equal marks.

1. (a) Find the general solution of the system of equations

$$\begin{aligned} x + 5y - 3z &= 7 \\ 3x + y + 7z &= 7 \\ 2x + 3y + 2z &= 7. \end{aligned}$$

(b) Sketch the feasible set given by the following inequalities, and find its vertices:

$$4x - 3y \geqslant 0, \qquad 2y - x \geqslant 1, \qquad 2x + 7y \geqslant 7.$$

(c) A triathlon competition consists of three events, running, swimming and cycling. The winner, running at 8 miles per hour, swimming at 2 miles per hour and cycling at 15 miles per hour, completed the course in 3 hours 35 minutes. One of the stragglers, running at 6 miles per hour, swimming at 1 mile per hour and cycling at 10 miles per hour, took 5 hours 40 minutes to complete the course. The total course was 32 miles long. How many miles was each segment (running, swimming, cycling)?

2. (a) What is meant by saying that a subset $S \subseteq \mathbb{R}^n$ is a *subspace*?

In each of the following cases, is S a subspace of $\mathbb{R}^3$? Give a full justification for each answer.

(*i*) $S = \{(a+b, a-b, a^2-b^2) : a, b \in \mathbb{R}\}$,
(*ii*) $S = \{(a+b, a-b, 0) : a, b \in \mathbb{R}\}$.

(b) For each of the following sets of vectors in $\mathbb{R}^3$, say whether or not it is linearly independent, giving reasons.

(*i*) $\{(1,2,3), (4,5,6)\}$,
(*ii*) $\{(1,2,3), (2,3,4), (3,4,5)\}$,
(*iii*) $\{(1,2,3), (2,3,1), (3,1,2), (3,2,1)\}$.

Question 2 continues . . .

(c) Define the *rank* of a matrix.

Find all the values of α for which rank $A < 3$, where

$$A = \begin{bmatrix} 1 & 1 & \alpha & 2\alpha^2 \\ -1 & \alpha & 1 & 1-3\alpha^2 \\ \alpha & -1 & -\alpha^2 & 2\alpha \end{bmatrix}.$$

3. (a) Find the inverse of the matrix $\begin{bmatrix} 2 & 2 & 1 \\ 3 & 5 & 4 \\ 5 & 6 & 4 \end{bmatrix}$.

Use your answer to solve the equations

$$\begin{aligned} 2x + 2y + z &= 7 \\ 3x + 5y + 4z &= 8 \\ 5x + 6y + 4z &= 9. \end{aligned}$$

What is the maximum amount by which each of x, y and z might change if the constants on the right hand side of the equations were altered by up to ± 0.5?

(b) Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 5 & 2 & 2 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}$.

Write down a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$.

4. (a) The economy of a small country depends on its output of coal and cement. To produce £1 worth of coal requires 8p of coal and 12p of cement. To produce £1 of cement uses 36p of coal and 4p of cement. How much coal and cement must be produced in order to meet a demand of £84 million for each of the two products?

(b) The Hungry Horse Eaterie chain of restaurants plans to open new restaurants, of three different sizes. A large restaurant requires an initial capital investment of £600 000, employs a staff of 15 and expects to make an annual profit of £40 000. A medium sized restaurant requires an initial investment of £400 000, has 9 employees and is expected to make an annual profit of £30 000. A small restaurant requires an initial investment of £300 000, employs 5 staff and generates an annual profit of £25 000. The Hungry Horse board of directors has £48 000 000 available for capital investment, but their corporate plan does not allow them to take on more than 1000 new employees, or to open more than 70 new restaurants. How many restaurants of each size should be opened to maximise the expected annual profit?

5. **(a)** Explain why the stochastic matrix $A = \begin{bmatrix} 1/4 & 1 & 1/4 \\ 0 & 0 & 1/2 \\ 3/4 & 0 & 1/4 \end{bmatrix}$ is regular. What is $\lim_{n\to\infty} A^n$?

(b) Explain why the stochastic matrix $B = \begin{bmatrix} 1 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & 1/6 \\ 0 & 0 & 2/3 & 0 \\ 0 & 0 & 1/3 & 1/6 \end{bmatrix}$ is absorbing. What is $\lim_{n\to\infty} B^n$?

(c) Determine the optimal strategy for R, for the game whose payoff matrix is

$$\begin{bmatrix} -3 & 4 \\ 2 & -2 \end{bmatrix}.$$

Which player does the game favour, and by how much?

END