MATH–133101

This question paper consists of 3 printed pages, each of which is identified by the reference MATH–1331

Only approved basic scientific calculators may be used.

Examination for the Module MATH–1331

(May/June 2002)

Linear algebra with applications

Time allowed : 3 hours

Answer **four** questions.
All questions carry equal marks.

1. **(a)** Use the Gaussian reduction process to find the complete solution set of the system of equations

$$\begin{aligned} x - 3y - 4z &= 0 \\ w + x + y + 3z &= 0 \\ 2w + 3x - y + 2z &= 0. \end{aligned}$$

Find a basis for the solution space.

(b) Sketch the feasible set given by the following set of inequalities, and find its vertices:

$$2x + y \leqslant 3, \qquad x + 5 \geqslant 5y, \qquad 4x \leqslant 3y, \qquad x \geqslant 0, \quad y \geqslant 0.$$

(c) The Beautiful Day Fruit Juice Company makes two varieties of fruit drink. Each can of Fruit Delight contains 10 ounces of pineapple juice, 3 ounces of orange juice and 1 ounce of apricot juice, and makes a profit of 20p. Each can of Heavenly Punch contains 10 ounces of pineapple juice, 2 ounces of orange juice and 2 ounces of apricot juice, and makes a profit of 30p. Each week, the company has available 9000 ounces of pineapple juice, 2400 ounces of orange juice and 1400 ounces of apricot juice. How many cans of Fruit Delight and of Heavenly Punch should be produced each week to maximise profits?

2. **(a)** Explain what is meant by a *linearly independent* set of vectors.

For each of the following sets of vectors in $\mathbb{R}^3$, find all values of s (if any exist) such that the set is linearly independent:

(i) $\{(1,0,1),\ (2,s,3),\ (2,3,1)\}$,

(ii) $\{(1,0,1),\ (2,s,3),\ (1,-s,0)\}$.

Question 2 continues . . .

(b) Find the dimension of the subspace of $\mathbb{R}^4$ spanned by each of the following sets of vectors:

(*i*) $\{(3,1,2,-1),\ (1,-3,2,-3),\ (1,2,0,1)\}$,

(*ii*) $\{(3,1,2,-1),\ (1,2,3,-2),\ (2,-1,1,3),\ (1,3,-1,2)\}$.

(c) Find the values of x for which $\begin{vmatrix} 1 & 2 & x \\ 2 & x & 8 \\ x & 8 & -4 \end{vmatrix} = 0.$

3. Throughout this question, A denotes the matrix $\begin{bmatrix} 2 & 3 & 3 \\ -3 & -4 & -3 \\ 2 & 2 & 1 \end{bmatrix}$.

(*i*) Find A^{-1}.

(*ii*) Solve the equations

$$\begin{aligned} 2x + 3y + 3z &= 1.2 \\ -3x - 4y - 3z &= -0.7 \\ 2x + 2y + z &= 2.6. \end{aligned}$$

(*iii*) Given that the constants in the above equations have been rounded to the nearest decimal place, find the maximum errors in the calculated values of x, y and z.

(*iv*) Find the eigenvalues and corresponding eigenvectors of A.

(*v*) Write down an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

4. (a) A manufacturing company has branches throughout Scotland, England and Wales. Its branches in one region purchase goods from branches elsewhere according to the matrix

$$\begin{array}{ll} & \text{for unit production in:} \\ & \begin{array}{ccc} \quad\mathbf{S} & \quad\mathbf{E} & \quad\mathbf{W} \end{array} \\ \text{Purchase from: } \begin{array}{l} \mathbf{S}\text{cotland} \\ \mathbf{E}\text{ngland} \\ \mathbf{W}\text{ales} \end{array} & \begin{bmatrix} 0.3 & 0 & 0.2 \\ 0.1 & 0.3 & 0.1 \\ 0.1 & 0 & 0.4 \end{bmatrix} \end{array} .$$

The external sales in each region are £6m in Scotland, £24m in England and £10m in Wales. At what rate should each region produce in order to satisfy the total demand?

(b) Explain what is meant by a simplex tableau.

(c) Use the simplex method to maximise $x + 6y + z$ subject to the constraints

$$x + 3y + 2z \leqslant 50, \quad 2x + y + 10 \leqslant z, \quad x \geqslant 0, \quad y \geqslant 0, \quad z \geqslant 0.$$

continued . . .

5. (a) Find the stable matrix for the Markov process whose transition matrix is

$$\begin{bmatrix} 1 & 0 & 3/8 & 0 & 1/8 \\ 0 & 1 & 1/8 & 1/8 & 1/8 \\ 0 & 0 & 1/4 & 1/8 & 1/4 \\ 0 & 0 & 0 & 3/8 & 0 \\ 0 & 0 & 1/4 & 3/8 & 1/2 \end{bmatrix}.$$

If the population is initially distributed equally among all five states, what percentage will eventually be in each of the two stable states?

(b) Determine the optimal strategy for R, for the game whose payoff matrix is

$$\begin{bmatrix} -2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

END