MATH–102201

This question paper consists of 2 printed pages, each of which is identified by the reference MATH–1022

Only approved basic scientific calculators may be used.

Examination for the Module MATH–1022

(May–June 2007)

Introductory Group Theory

Time allowed : 2 hours

Answer not more than **four** questions. All questions carry equal marks.

1. **(a)** Give the definition of a *group*.

(b) Give the definition of an *abelian* group.

(c) Let $G = \{3, 6, 9, 12\}$ under multiplication mod 15. Prove that G is a group.

(d) Prove that the set of non-zero real numbers under division is **not** a group.

(e) Prove that in every group, the identity element is unique.

2. Let G denote the group of symmetries (rotations and reflections) of a regular hexagon.

(a) List the elements of G, and find the order of each of the elements of G.

(b) Let $a \in G$ denote a rotation through $2\pi/6$ clockwise.
Find the subgroup H of G generated by a.

(c) Consider the group $\mathbb{Z}_9^* = \{1, 2, 4, 5, 7, 8\}$ under $\times$ mod 9.
Prove that $\{1, 4, 7\}$ is a subgroup of $\mathbb{Z}_9^*$ and give an example of a non-empty subset of $\mathbb{Z}_9^*$ which is **not** a subgroup of $\mathbb{Z}_9^*$.

3. **(a)** Explain what it means to say that two groups G_1 and G_2 are *isomorphic*.

(b) Let $G = \{e, a, b, c\}$ be a group in which the square of each element of G is the identity element, e. Using the Latin Square property of G, or otherwise, find the Cayley table of G.

(c) Prove that $\mathbb{Z}_7^*$, under $\times$ mod 7, is cyclic, and hence show that it is isomorphic to the group $\mathbb{Z}_6$ under $+$ mod 6. Give an example of a group of order 6 not isomorphic to either of these groups.

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4. Let G be a group and let H be a subgroup of G.

(a) Let $x, y \in G$. Prove that $Hx = Hy$ if and only if $xy^{-1} \in H$.

(b) Let $x \in G$. Prove that if Hx is a subgroup of G, then $x \in H$.

(c) Suppose that $G = \mathbb{Z}_{13} \setminus \{0\}$, under $\times$ mod 13. Find all of the right cosets of the subgroup $H = \{1, 3, 9\}$ of G. Find two different elements, $x, y \in G$, such that $Hx = Hy$.

5. **(a)** Write the following product of permutations as a product of disjoint cycles:

$$(1\ 7\ 3\ 4)(3\ 5\ 4)(1\ 6)(2\ 3\ 7).$$

(b) Write the permutation given below (in two-line notation) as a product of disjoint cycles, and compute its order, stating carefully any results you use:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 9 & 6 & 8 & 3 & 7 & 5 & 1 & 2 \end{pmatrix}.$$

(c) Give the definition of the *alternating group* of degree n and prove that it is a subgroup of the symmetric group of degree n.

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