MATH–102201

This question paper consists of 2 printed pages, each of which is identified by the reference MATH–1022

Only approved basic scientific calculators may be used.

Examination for the Module MATH–1022
(May–June 2006)

Introductory Group Theory
Time allowed : 2 hours

Answer not more than **four** questions. All questions carry equal marks.

1. (a) Determine which of the following are groups. For those which are, state the identity and inverses. For those which are not, give one axiom that fails.

(i) $\mathbb{N} = \{1, 2, 3, \ldots\}$ under $+$.

(ii) $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ under $+$.

(iii) $\{1, 2, 3\}$ under $\times$ mod 4.

(iv) The set of $n \times n$ matrices with determinant 1 under matrix multiplication.

(b) Consider the group $\mathbb{Z}_{12}^* = \{1, 5, 7, 11\}$ under $\times$ mod 12. Prove that $\{1, 11\}$ is a subgroup, but $\{1, 7, 11\}$ is not.

2. (a) Let G and H be groups. Define the *direct product* $G \times H$ and prove that it is a group.

(b) If G and H are finite groups, give a formula for the order $|G \times H|$ in terms of the orders $|G|$ and $|H|$.

(c) Determine which of the following groups are cyclic, giving reasons for your answers: $\mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, and $\mathbb{Z}_2 \times \mathbb{Z}_5$.

3. (a) Consider the group $\mathbb{Z}_4$ under $+$ mod 4. Write down an isomorphism between this group and the group with group table

	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

Question 3 continues . . .

(b) State Lagrange's Theorem and use it to prove that if g is an element of a finite group G, then $|g|$, the order of the element g, divides $|G|$, the order of the group G.

(c) Consider the group $\mathbb{Z}_{12}$ under + mod 12. The subset $H = \{0, 6\}$ is a subgroup. Determine the right cosets of H in $\mathbb{Z}_{12}$.

4. **(a)** Consider the cycle (1 6 3 5 4 2) in S_6. Write it as a composition of transpositions.

(b) Let

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 8 & 1 & 3 & 7 & 2 & 6 \end{pmatrix} \text{ and } g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 8 & 6 & 2 & 3 & 1 & 7 & 4 \end{pmatrix}$$

be elements of the symmetric group S_8, given in two line notation. Write the composition gf in two line notation. Write f, g, gf, and $(gf)^2$ as products of disjoint cycles.

(c) Determine which of the permutations f, g, gf, and $(gf)^2$ are even and which are odd.

5. **(a)** Let G and H be groups. Define what it means for a map $\theta : G \to H$ to be a *homomorphism*. Define the *kernel* of θ and prove that it is a normal subgroup of G.

(b) Determine which of the following are homomorphisms, giving reasons for your answers. For each homomorphism, determine the kernel.

(i) $\theta(x) = x^2$ from $\{ x \in \mathbb{R} \mid x \neq 0 \}$ under $\times$ to itself.

(ii) $\theta(x) = x + x$ from $\{ x \in \mathbb{R} \mid x \neq 0 \}$ under $\times$ to itself.

(iii) $\theta(x) = x + x$ from $\mathbb{R}$ under + to itself.

(iv) $\theta(x) = x + x$ from $\mathbb{Z}_{12}$ under + mod 12 to itself.

END