MATH–102201

This question paper consists of 2 printed pages, each of which is identified by the reference MATH–1022

Only approved basic scientific calculators may be used.

Examination for the Module MATH–1022
(May 2005)

Introductory Group Theory
Time allowed : 2 hours

Answer not more than **four** questions. All questions carry equal marks.

1. (i) Determine which of the following are groups. For those which are not groups, give one axiom that fails. For those which are groups, find the identity and inverses.

   (a) $\{1, 3, 5, 7, 9\}$ under multiplication mod 10,

   (b) $\{3, 6, 9, 12\}$ under multiplication mod 15,

   (c) the set of positive rational numbers under multiplication,

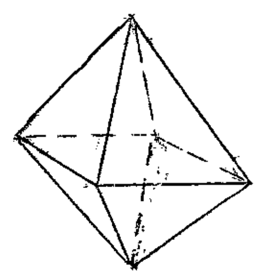
   (d) the set of 2×2 matrices with integer entries, under matrix subtraction.

(ii) Prove that for any elements g and h of a group,

$$(gh)^{-1} = h^{-1}g^{-1} \text{ and } (g^{-1})^{-1} = g.$$

(iii) Prove that a group G is abelian if and only if for every g and h in G, $(gh)^2 = g^2h^2$.

2. (i) A *regular octahedron* is a solid with eight faces, all congruent equilateral triangles, six vertices and twelve edges (see the diagram). Explain why the group of rotations OCT of a regular octahedron has order 24. Find how many elements of OCT there are of each of the following kinds:



identity,
rotation about a line joining opposite vertices,
rotation about a line joining the midpoints of opposite faces,
rotation about a line joining the midpoints of opposite edges,

and check that these numbers add to 24. Find the orders of all elements of OCT.

(ii) Given that just one of the following groups is not isomorphic to the other two, determine which one it is, and explain why:

   (a) $\{1, 3, 7, 9, 11, 13, 17, 19\}$ under $\times$ mod 20,

   (b) $\mathbb{Z}_4 \times \mathbb{Z}_2$,

   (c) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

3. (i) Prove Lagrange's Theorem, that the order of a subgroup H of a finite group G divides the order of G. (Hint: first show that $\sim$ given by $x \sim y$ if $xy^{-1} \in H$ is an equivalence relation on G.)

(ii) Prove that $H = \{1, 7, 11\}$ is a subgroup of $G = \{1, 2, 3, \ldots, 18\}$ under multiplication mod 19. Find all the right cosets of H in G. What is the index of H in G? What can you say about the left cosets of H in G?

4. (i) Prove that the set of all permutations of a set X forms a group under function composition. Find the order of this group when X has 3, 4, 5 elements.

(ii) Let $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 8 & 6 & 10 & 4 & 1 & 2 & 5 & 9 & 7 \end{pmatrix}$ and $g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 4 & 5 & 1 & 7 & 8 & 9 & 10 & 6 & 2 & 3 \end{pmatrix}$ be elements of S_{10}. Write each of the following as a product of disjoint cycles:

f, g, fg, gf, f^2, g^3.

(iii) Show that a cycle of odd length is an even permutation, and a cycle of even length is an odd permutation, and hence determine which of the permutations in part (ii) are even.

5. (i) The group table of the dihedral group D_4 of order 8 is given.

	I	R	R^2	R^3	H	V	D	D'
I	I	R	R^2	R^3	H	V	D	D'
R	R	R^2	R^3	I	D'	D	H	V
R^2	R^2	R^3	I	R	V	H	D'	D
R^3	R^3	I	R	R^2	D	D'	V	H
H	H	D	V	D'	I	R^2	R	R^3
V	V	D'	H	D	R^2	I	R^3	R
D	D	V	D'	H	R^3	R	I	R^2
D'	D'	H	D	V	R	R^3	R^2	I

Which of the following subgroups are normal? $\{I, D\}$, $\{I, D, D', R^2\}$. Give reasons.

(ii) Find the right cosets of the normal subgroup $N = \{I, R^2\}$, and draw up the group table for the quotient group D_4/N. Is this quotient group cyclic?

(iii) Describe a homomorphism θ from D_4 onto D_4/N. What is the kernel of θ?

END