MATH–102201

This question paper consists of 2 printed pages, each of which is identified by the reference MATH–102201

Only approved basic scientific calculators may be used

Examination for the Module MATH–102201

(May/June, 2004)

Introductory Group Theory

Time allowed : 2 hours

Answer **four** questions. All questions carry equal marks.

1. (i) Define *group* and *abelian group*.

(ii) Determine which of the following are groups. For those which are groups, state the identity and inverses, and whether or not they are abelian. For those which are not groups, give one axiom that fails:

(a) the set of real numbers under subtraction,
(b) the set of complex numbers of modulus 1, under addition,
(c) the set of 2×2 real matrices of determinant 1, under matrix multiplication,
(d) the set $\{0, 1, 2, 3, 4\}$ under the operation given in the table:

	0	1	2	3	4
0	2	3	4	0	1
1	3	4	0	1	2
2	4	0	1	2	3
3	0	1	2	3	4
4	1	2	3	4	0

(iii) Prove the 'Latin square property' of a group table, that each element of a group occurs exactly once in each row, and exactly once in each column. (You may assume without proof the 'cancellation laws', but should state them clearly.)

2. (i) Let CUB be the group of all rotations of a cube. Explain why CUB has order 24. Describe all elements of CUB, and state their orders. (You should state what the possible axes of rotation are, and for each of these, what angles you can rotate through.)

(ii) Define *cyclic group*, and prove that any two cyclic groups of the same order are isomorphic.

(iii) Determine, giving reasons, which of the following groups are cyclic:

(a) the group of even integers, under addition,
(b) the group of real numbers, under addition,
(c) $\mathbb{Z}_6 \times \mathbb{Z}_{10}$,
(d) $\mathbb{Z}_8 \times \mathbb{Z}_{11}$.

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3. (i) Define *left coset* and *right coset* of a subgroup H of a group G. The group table of the 'dihedral group' D_5 is as follows:

	1	a	a^2	a^3	a^4	b	ab	a^2b	a^3b	a^4b
1	1	a	a^2	a^3	a^4	b	ab	a^2b	a^3b	a^4b
a	a	a^2	a^3	a^4	1	ab	a^2b	a^3b	a^4b	b
a^2	a^2	a^3	a^4	1	a	a^2b	a^3b	a^4b	b	ab
a^3	a^3	a^4	1	a	a^2	a^3b	a^4b	b	ab	a^2b
a^4	a^4	1	a	a^2	a^3	a^4b	b	ab	a^2b	a^3b
b	b	a^4b	a^3b	a^2b	ab	1	a^4	a^3	a^2	a
ab	ab	b	a^4b	a^3b	a^2b	a	1	a^4	a^3	a^2
a^2b	a^2b	ab	b	a^4b	a^3b	a^2	a	1	a^4	a^3
a^3b	a^3b	a^2b	ab	b	a^4b	a^3	a^2	a	1	a^4
a^4b	a^4b	a^3b	a^2b	ab	b	a^4	a^3	a^2	a	1

Find all left and right cosets of the subgroup $H = \{1, b\}$. Is H a normal subgroup of D_5?

(ii) Prove that the order of any element of a finite group divides the order of the group. (You may assume Lagrange's theorom, but should state it clearly.)

(iii) Deduce that if p is a prime number and p does not divide a then $a^{p-1} \equiv 1 \bmod p$.

4. (i) Define *even* and *odd permutation* of a finite set X. Prove that the set of all even permutations of X forms a subgroup (the 'alternating group') of the group of all permutations of X. List all the members of the alternating groups $\mathcal{A}_3$ and $\mathcal{A}_4$ (corresponding to $X = \{1, 2, 3\}$ and $X = \{1, 2, 3, 4\}$ respectively).

(ii) Write the permutations $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 5 & 6 & 7 & 8 & 9 & 1 & 2 & 3 \end{pmatrix}$ and $g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 9 & 4 & 5 & 6 & 7 & 3 & 1 & 2 \end{pmatrix}$ as products of disjoint cycles. Evaluate the products fg, gf, fg^{-1}, and show that f, g, and hence fg, gf, fg^{-1} are even.

(iii) State a criterion for conjugacy in $\mathcal{S}_n$, and deduce which of f, g, fg, gf, fg^{-1} are conjugate.

5. (i) Define *homomorphism* from a group G to a group H. Prove that if θ is a homomorphism from G to H, then the kernel $K = \{g \in G : \theta(g) = 1\}$ is a normal subgroup of G, and the range of θ is a subgroup of H.

(ii) Show that if $\mathbb{Z}_9$ is the group $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ under addition mod 9, then θ and φ given by $\theta(x) = x + x$ and $\varphi(x) = x + x + x$ are homomorphisms from $\mathbb{Z}_9$ to $\mathbb{Z}_9$. Show that θ is 1–1 but that φ is not. Find the kernel and range of φ.

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