MATH–102201

This question paper consists of 2 printed pages, each of which is identified by the reference MATH–1022

Only approved basic scientific calculators may be used.

Examination for the Module MATH–1022

(May/June, 2003)

Introductory Group Theory

Time allowed : 2 hours

Answer **four** questions. All questions carry equal marks

1. **(i)** Determine which of the following are groups. For those which are, state the identity and inverses. For those which are not, give one axiom that fails:

(a) the set $\{\ldots, -\frac{5}{2}, -2, -\frac{3}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots\}$ of rational numbers of the form $a/2$ for a an integer, under addition,
(b) $\{2, 4, 6, 8, 10\}$ under multiplication mod 12,
(c) $\{1, 3, 5, 9, 11, 13\}$ under multiplication mod 14,
(d) the set $\{0, 1, 2, 3\}$ under the operation given in the table:

	0	1	2	3
0	2	0	3	1
1	1	3	0	2
2	0	1	2	3
3	3	2	1	0

(ii) Prove that for any prime number n, $\{1, 2, 3, \ldots, n-1\}$ forms a group under multiplication mod n. [You may assume without proof that if m and n are coprime then there are integers x and y such that $mx + ny = 1$.] Calculate the order of this group when $n = 19$, and find a subgroup of order 3

2. **(i)** Define the *direct product* $G \times H$ of two groups G and H, and prove that $G \times H$ is a group. Show further that $G \times H$ is abelian if, and only if, both G and H are abelian

(ii) Show that $\mathbb{Z}_8, \mathbb{Z}_4 \times \mathbb{Z}_2$, and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ are all abelian groups of order 8, and by considering the orders of their elements, or otherwise, show that no two of them are isomorphic.

Given that the group $\{1, 3, 7, 9, 11, 13, 17, 19\}$ under $\times$ mod 20 is isomorphic to one of the three given groups, determine which one it is

(iii) Describe a non-abelian group of order 8 having exactly two elements of order 4, and find the orders of its other elements

continued . . .

3. (i) Determine which of the following four groups are isomorphic, giving reasons.

(a) $\{1,2,3,4\}$ under multiplication mod 5,
(b) the set of integers in $\{1,2,\ldots,8\}$ coprime with 9 under multiplication mod 9,
(c) the set of integers in $\{1,2,\ldots,11\}$ coprime with 12 under multiplication mod 12,
(d) the group of symmetries of a (non-square) rectangle.

(ii) Find all right and left cosets of the subgroup $\{I, C\}$ of the dihedral group D_3 of order 6, whose table is given

	I	R	S	A	B	C
I	I	R	S	A	B	C
R	R	S	I	B	C	A
S	S	I	R	C	A	B
A	A	C	B	I	S	R
B	B	A	C	R	I	S
C	C	B	A	S	R	I

(iii) Prove Fermat's Little Theorem, that if p is a prime number, then for any integer a, $a^p \equiv a \bmod p$ [Any standard results of group theory used should be clearly stated.] Hence calculate the remainder on dividing 3^{203} by 101, justifying your method.

4. (i) Define *permutation* of a set X Prove that any permutation of a finite set can be written as a product of disjoint cycles

(ii) For the permutations $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 4 & 7 & 5 & 2 & 8 & 1 \end{pmatrix}$ and $g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 6 & 7 & 4 & 8 & 5 & 1 \end{pmatrix}$ of $\{1,2,3,4,5,6,7,8\}$, write each of f, g, fg, gf and fg^{-1} as a product of disjoint cycles, and determine which of them are even.

(iii) Describe all conjugacy classes of S_4, the group of all permutations of $\{1,2,3,4\}$, quoting clearly any standard results needed to justify this. Find the order of S_4 and the number of elements in each conjagacy class.

5. (i) The group table of the quaternion group Q is given Which of the following subgroups are normal? (a) $\{1, a, a^2, a^3\}$; (b) $\{1, b, a^2, a^2b\}$. Give reasons.

	1	a	a^2	a^3	b	ab	a^2b	a^3b
1	1	a	a^2	a^3	b	ab	a^2b	a^3b
a	a	a^2	a^3	1	ab	a^2b	a^3b	b
a^2	a^2	a^3	1	a	a^2b	a^3b	b	ab
a^3	a^3	1	a	a^2	a^3b	b	ab	a^2b
b	b	a^3b	a^2b	ab	a^2	a	1	a^3
ab	ab	b	a^3b	a^2b	a^3	a^2	a	1
a^2b	a^2b	ab	b	a^3b	1	a^3	a^2	a
a^3b	a^3b	a^2b	ab	b	a	1	a^3	a^2

(ii) Find all the right cosets of the normal subgroup $N = \{1, a^2\}$ of Q, and draw up the group table for the quotient group Q/N Is this group cyclic?

(iii) Describe a homomorphism θ from Q onto Q/N What is the kernel of θ?

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