MATH–011101

This question paper consists of 5 printed pages, each of which is identified by the reference MATH–011101

Only approved basic scientific calculators may be used.

Examination for the Module MATH–0111

(January 2003)

Elementary Differential Calculus (Version 1)

Time allowed: 2 hours

Attempt **all** the questions in Section A and **three** questions from Section B.

Each question in Section A carries 2 marks, and each question in Section B carries 20 marks.
You must show your working in answers to all questions.
A formula sheet is supplied with this paper.

SECTION A

Attempt **all** the questions in Section A

A1. Expand $(2x-3)(x+5)$.

A2. Evaluate $(27)^{-2/3}$.

A3. Evaluate $2a^3ba^{-1}b^{-1/2}$ when $a=5$ and $b=9$.

A4. Find $\log_4 32$.

A5. Factorise $x^2+3x-10$.

A6. Solve the equation $x^2+5x=0$.

A7. Find the equation of the straight line through the point $(-1,1)$ which is parallel to the line $2x-3y+1=0$.

A8. What is the distance between the points $(1,2)$ and $(-2,-2)$?

A9. The angle θ lies between 0 and $\pi/2$, and $\cos\theta = 2/5$. Find $\sin\theta$ and $\tan\theta$, leaving your answers as exect expressions involving square roots.

A10. Find the equation of the circle with centre $(3,0)$ and radius 4.

A11. Find $\dfrac{dy}{dx}$ when $y=x^{4/3}$.

continued ...

A12. Find $\dfrac{dy}{dx}$ when $y = 3x^2 + 4x + 7$.

A13. Find $\dfrac{dy}{dx}$ when $y = \sqrt{3x - 1}$.

A14. Find $\dfrac{dy}{dx}$ when $y = \cos^4 x$.

A15. Find $\dfrac{dy}{dx}$ when $y = \dfrac{x+1}{x^2+1}$.

A16. Find $\dfrac{dy}{dx}$ when $y = e^{2x} \sin x$.

A17. Find $\dfrac{dy}{dx}$ when $y = \ln(x^3 + 4)$.

A18. Find $\dfrac{d^2y}{dx^2}$ when $y = 3x^4 - 5x^2$.

A19. Find the tangent to the curve $y = 2x^2 - x + 3$ at the point $(1, 4)$.

A20. Without using a calculator, find an exact expression for $\cos(2\pi/3)$.

SECTION B

Attempt **three** questions in Section B

B1. **(a)** Sketch the graph of $y = \sin\theta$, for θ in the range $-2\pi \leqslant \theta \leqslant 2\pi$, labelling the values of θ where the graph crosses the horizontal axis and where y has minimum or maximum values.

(b) Find all values of θ (in radians) between -2π and 2π, such that $\sin\theta = -1/2$.

(c) Using the formula for $\sin(A + B)$ from the formula sheet, show that $\sin(\theta + \frac{\pi}{2}) = \cos\theta$. Use the results from the previous part of the question to find all values of θ between $-\pi$ and π such that $\cos\theta = -1/2$.

B2. **(a)** The points A, B have coordinates $(1, 4)$ and $(3, 3)$. Find

(i) the equation of the line AB;

(ii) the equation of the line through the origin perpendicular to AB;

(iii) the point where the above two lines meet;

(iv) the distance from the origin to the line AB.

Question B2 continues . . .

(b) A circle has centre at the point $C = (2, 3)$ and passes through the point $P = (4, 0)$. Find

(i) the radius of the circle;

(ii) the equation of the circle;

(iii) the gradient of the line CP;

(iv) the equation of the tangent to the circle at P.

B3. Differentiate each of the following functions with respect to x.

(i) $y = (x^3 - 4)^4 + (x^3 - 4)^{-1/2}$;

(ii) $y = (2x^2 + 3x)\tan x$;

(iii) $y = \dfrac{\ln x}{x(x+1)}$;

(iv) $y = \arcsin(x^2)$;

(v) $y = (e^{3x} + 2)^7$.

B4. **(a)** Find the stationary points of the function given by $y = x^3 - 3x^2 - 9x$ and determine whether they are (local) maximum or minimum points.

(b) Find the maximum and minimum values of $1 + 4x - x^2$ for x between 0 and 5.

(c) If y is given as a function of x by $x^2y + 2y^3 = 5x - 2$, find $\dfrac{dy}{dx}$ in terms of x and y.

END

Elementary Differential and Integral Calculus

FORMULA SHEET

Exponents

$x^a \cdot x^b = x^{a+b}, \quad a^x \cdot b^x = (ab)^x, \quad (x^a)^b = x^{ab}, \quad x^0 = 1.$

Logarithms

$\ln xy = \ln x + \ln y, \quad \ln x^a = a \ln x, \quad \ln 1 = 0, \quad e^{\ln x} = x, \quad \ln e^y = y,$
$a^x = e^{x \ln a}.$

Trigonometry

$\cos 0 = \sin \frac{\pi}{2} = 1, \qquad \sin 0 = \cos \frac{\pi}{2} = 0,$
$\cos^2\theta + \sin^2\theta = 1, \qquad \cos(-\theta) = \cos\theta, \qquad \sin(-\theta) = -\sin\theta,$
$\cos(A+B) = \cos A \cos B - \sin A \sin B, \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta,$
$\sin(A+B) = \sin A \cos B + \cos A \sin B, \qquad \sin 2\theta = 2 \sin\theta \cos\theta,$
$\tan\theta = \dfrac{\sin\theta}{\cos\theta}, \qquad \sec\theta = \dfrac{1}{\cos\theta}, \qquad 1 + \tan^2\theta = \sec^2\theta.$

Inverse Functions

$y = \sin^{-1} x$ means $x = \sin y$ and $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$.
$y = \cos^{-1} x$ means $x = \cos y$ and $0 \leqslant y \leqslant \pi$.
$y = \tan^{-1} x$ means $x = \tan y$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$.
$y = x^{1/n}$ means $x = y^n$. $\qquad y = \ln x$ means $x = e^y$.

Alternative Notation

$\arcsin x = \sin^{-1} x, \quad \arccos x = \cos^{-1} x, \quad \arctan x = \tan^{-1} x, \quad \log_e x = \ln x.$
Note: $\sin^{-1} x \neq (\sin x)^{-1}, \quad \cos^{-1} x \neq (\cos x)^{-1}, \quad \tan^{-1} x \neq (\tan x)^{-1}.$
However: $\sin^2 x = (\sin x)^2, \quad \cos^2 x = (\cos x)^2, \quad \tan^2 x = (\tan x)^2.$

Lines

The line $y = mx + c$ has slope m.
The line through (x_1, y_1) with slope m has equation $y - y_1 = m(x - x_1)$.
The line through (x_1, y_1) and (x_2, y_2) has slope $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ and equation $\dfrac{y - y_1}{x - x_1} = \dfrac{y_2 - y_1}{x_2 - x_1}$.

The line $y = mx + c$ is perpendicular to the line $y = m'x + c'$ if $mm' = -1$.

Circles

The distance between (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.
The circle with centre (a, b) and radius r is given by $(x - a)^2 + (y - b)^2 = r^2$.

Triangles

In a triangle ABC,

$$\text{(Sine Rule)} \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}; \qquad \text{(Cosine Rule)} \quad a^2 = b^2 + c^2 - 2bc \cos A.$$

Pascal's Triangle

$(x+y)^2 = x^2 + 2xy + y^2, \quad (x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$ and so on.

The coefficients in $(x+y)^n$ form the nth row of Pascal's triangle:

$$\begin{array}{ccccccccc} & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

and so on.

Quadratics

If $ax^2 + bx + c = 0$, with $a \neq 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Calculus

If $y = u + v$ then $\dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$. If $y = uv$ then $\dfrac{dy}{dx} = \dfrac{du}{dx}v + u\dfrac{dv}{dx}$.

If $y = \dfrac{u}{v}$ then $\dfrac{dy}{dx} = \left\{\dfrac{du}{dx}v - u\dfrac{dv}{dx}\right\} \Big/ v^2$.

$\displaystyle\int (u+v)\,dx = \int u\,dx + \int v\,dx.$ $\displaystyle\int u\frac{dv}{dx}\,dx = uv - \int \frac{du}{dx}v\,dx.$

If y is a function of u where u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \quad \text{and} \quad \int y\frac{du}{dx}\,dx = \int y\,du.$$

Standard Derivatives and Integrals

If $y = x^a$ then $\dfrac{dy}{dx} = a\,x^{a-1}$; and $\displaystyle\int x^a\,dx = \frac{x^{a+1}}{a+1} +$ constant $(a \neq -1)$.

If $y = \sin x$ then $\dfrac{dy}{dx} = \cos x$; and $\displaystyle\int \sin x\,dx = -\cos x +$ constant.

If $y = \cos x$ then $\dfrac{dy}{dx} = -\sin x$; and $\displaystyle\int \cos x\,dx = \sin x +$ constant.

If $y = \tan x$ then $\dfrac{dy}{dx} = \sec^2 x$; and $\displaystyle\int \tan x\,dx = \ln|\sec x| +$ constant.

If $y = e^x$ then $\dfrac{dy}{dx} = e^x$; and $\displaystyle\int e^x\,dx = e^x +$ constant.

If $y = \ln x$ then $\dfrac{dy}{dx} = \dfrac{1}{x}$; and $\displaystyle\int \frac{1}{x}\,dx = \ln|x| +$ constant.

If $y = \sin^{-1} x$ then $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}}$; and $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x +$ constant.

If $y = \cos^{-1} x$ then $\dfrac{dy}{dx} = \dfrac{-1}{\sqrt{1-x^2}}$.

If $y = \tan^{-1} x$ then $\dfrac{dy}{dx} = \dfrac{1}{1+x^2}$; and $\displaystyle\int \frac{1}{1+x^2}\,dx = \tan^{-1} x +$ constant.